Honors Mathematics B Lecture notes

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Contents

1 Inner product spaces **3**

1.1 Inner product . . . 3

1.2 Norm and distance . . . 6

1.3 Orthogonality . . . 8

1.4 Bases and projections . . . 14

1.5 Riesz representation theorem . . . 17

2 Metric spaces **19**

2.1 Introduction . . . 19

2.2 Open and closed sets . . . 20

2.3 Convergence in a metric space . . . 21

2.4 The closure of a set . . . 22

2.5 Dense subsets . . . 22

2.6 Continuity . . . 24

2.7 Completeness . . . 25

3 Hilbert spaces **30**

3.1 Infinite series . . . 31

3.2 Best approximation . . . 33

3.3 Fourier expansions . . . 37

3.3.1 The finite-dimensional case . . . 37

3.3.2 The countably infinite-dimensional case 38
3.3.3 The uncountably infinite-dimensional case 39
3.4 Characterization of Hilbert spaces 42
3.5 The Riesz representation theorem 44

4 Fourier series **47**

4.1 Computation of Fourier series 48
4.2 Convergence of Fourier series 52
4.3 Solving the heat equation 58

1 Inner product spaces

1.1 Inner product

Let us first review the definition of a vector space.

Definition 1.1 (Field)**.** *A* **field** *is a set F, containing at least two elements (referred to as* **scalars***), on which two operations $+$ and $\cdot$ (called addition and multiplication, respectively) are defined so that for each pair of elements $a, b \in F$ there are unique elements $a+b$ and $a \cdot b$ (often written ab) in F for which the following conditions hold for all elements $a, b, c \in F$:*

(i) (***Commutativity of addition***) $a + b = b + a$

(ii) (***Associativity of addition***) $(a + b) + c = a + (b + c)$

(iii) (***Existence of an additive identity***) *There is an element $0 \in F$, called zero, such that $a + 0 = a$.*

(iv) (***Existence of additive inverses***) *For each a, there is an element $-a \in F$ such that $a + (-a) = 0$.*

(v) (***Commutativity of multiplication***) $ab = ba$

(vi) (***Associativity of multiplication***) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(vii) (***Distributivity***) $(a + b) \cdot c = a \cdot c + b \cdot c$ *and* $a \cdot (b + c) = a \cdot b + a \cdot c$

(viii) (***Existence of a multiplicative identity***) *There is an element $1 \in F$, such that $1 \neq 0$ and $a \cdot 1 = a$.*

(ix) (***Existence of multiplicative inverses***) *If $a \neq 0$, then there is an element $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.*

Example Sets of rational numbers $\mathbb{Q}$, set of real numbers $\mathbb{R}$ and set of complex numbers $\mathbb{C}$ are all fields. However, set of integers $\mathbb{Z}$ is not a field.

Definition 1.2 (Vector space)**.** *A* **vector space** *over a field F is a nonempty set V, whose elements are referred to as* **vectors***, together with two operations: addition (denoted by $+$) and scalar multiplication (denoted by juxtaposition). Furthermore, the following properties must be satisfied:*

(1) (***Associativity of addition***) *For all vectors $u, v, w \in V$*

$$u + (v + w) = (u + v) + w$$

(2) (***Commutativity of addition***) *For all vectors $u, v \in V$*

$$u + v = v + u$$

(3) (***Existence of a zero***) *There is a vector $0 \in V$ with the property that*

$$0 + u = u + 0 = u \qquad \text{for all vectors } u \in V$$

(4) (***Existence of additive inverses***) *For each vector $u \in V$, there is a vector in V, denoted by $-u$, with the property that*

$$u + (-u) = (-u) + u = 0$$

(5) ***(Properties of scalar multiplication)*** *For all scalars $a, b \in F$ and for all vectors $u, v \in V$,*

$$\begin{aligned} a(u+v) &= au + av \\ (a+b)u &= au + bv \\ (ab)u &= a(bu) \\ 1u &= u \end{aligned}$$

Example Classical examples of vector spaces:

(1) Set of all functions from a field F to F is a vector space over F (with the usual operations of addition and scalar multiplication of functions).

(2) The set $\mathcal{M}_{m,n}(F)$ of all $m \times n$ matrices with entries in a field F is a vector space over F (under usual matrix addition and scalar multiplication).

(3) The set of all ordered n-tuples whose components lie in a field F, is a vector space over F (with addition and scalar multiplication defined componentwise). Well-known example: $\mathbb{R}^n$.

(4) The set of all infinite sequences with members from a field F is a vector space over F (with addition and scalar multiplication defined componentwise). Other examples of sequence spaces that often appear are:
- set c_0 of all sequences of complex numbers that converge to zero
- set ℓ^∞ of all bounded complex sequences
- (for a positive integer p) set ℓ^p of all complex sequences $\{s_n\}$ for which

$$\sum_{n=1}^{\infty} |s_n|^p < \infty$$

If another function (called inner product) can be defined on a vector space, many more interesting things can be said about the space. In this chapter, we will consider only vector spaces V over the fields $F = \mathbb{R}$ or $F = \mathbb{C}$.

Definition 1.3 (Inner product)**.** *An* **inner product** *on V is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ with the following properties:*

(1) ***(Positive definiteness)*** *For all $v \in V$*

$$\langle v, v \rangle \geq 0 \qquad \text{and} \qquad \langle v, v \rangle = 0 \;\Leftrightarrow\; v = 0$$

(2) ***(Conjugate symmetry)***

$$\langle u, v \rangle = \overline{\langle v, u \rangle},$$

where bar denotes complex conjugate. For $F = \mathbb{R}$ this simply means the symmetry $\langle u, v \rangle = \langle v, u \rangle$.

(3) ***(Linearity in first coordinate)*** *For all $u, v \in V$ and $r, s \in F$*

$$\langle ru + sv, w \rangle = r\langle u, w \rangle + s\langle v, w \rangle$$

A real or complex vector space V which has an inner product is called a **real (complex) inner product space**.

Remark 1. A vector subspace S of an inner product space V is also an inner product space under the restriction of the inner product of V to S. (**HW**(1p): Prove it.)

2. The quantity $\langle v, v\rangle$ is always real, even if V is a complex vector space. (**HW**(1p): Prove it.)

3. If $F = \mathbb{R}$, then the inner product is linear in both coordinates, that is, it is **bilinear**. If $F = \mathbb{C}$, then it is **sesquilinear**, which means that it is linear in the first coordinate and **conjugate linear** in the second coordinate:

$$\langle w, ru + sv\rangle = \overline{\langle ru + sv, w\rangle} = \overline{r}\overline{\langle u, w\rangle} + \overline{s}\overline{\langle v, w\rangle} = \overline{r}\langle w, u\rangle + \overline{s}\langle w, v\rangle.$$

Example We present several examples of inner product spaces.

(1) n-dimensional Euclidean space: the vector space $\mathbb{R}^n$ with the standard inner product

$$\langle (r_1, \dots, r_n), (s_1, \dots, s_n)\rangle = r_1 s_1 + \cdots + r_n s_n.$$

(2) n-dimensional unitary space: the vector space $\mathbb{C}^n$ with the standard inner product

$$\langle (r_1, \dots, r_n), (s_1, \dots, r_n)\rangle = r_1 \overline{s}_1 + \cdots + r_n \overline{s}_n.$$

(3) The vector space $C[a, b]$ of all continuous complex-valued functions on the closed interval $[a, b]$ with the inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

(4) The vector space ℓ^2 of all real (or complex) sequences $\{s_n\}$ which are "square summable", that is,

$$\sum_{n=0}^{\infty} |s_n|^2 < \infty,$$

under the inner product

$$\langle \{s_n\}, \{t_n\}\rangle = \sum_{n=0}^{\infty} s_n \overline{t}_n.$$

The sum on the right-hand side converges as we can see by rearranging the inequality $0 \leq (|s_n| - |t_n|)^2$ to get

$$|s_n t_n| \leq \frac{1}{2}|s_n|^2 + \frac{1}{2}|t_n|^2.$$

(**HW**(1p): Verify that ℓ^2 is an inner product space.

Lemma 1.4. *If V is an inner product space and $\langle u, x\rangle = \langle v, x\rangle$ for all $x \in V$, then $u = v$.*

Proof. **HW**(1p). □

Definition 1.5 (Linear operator)**.** *Let V be a vector space over a field F. A function $T : V \to V$ is a* **linear operator** *if*

$$T(ru + sv) = rT(u) + sT(v)$$

for all scalars $r, s \in F$ and vectors $u, v \in V$. The set of of all linear operators on V is denoted by $\mathcal{L}(V)$.

Theorem 1.6. *Let V be an inner product space and let $T \in \mathcal{L}(V)$. Then*

(1) $\langle Tv, w\rangle = 0$ *for all* $v, w \in V \quad \Rightarrow \quad T = 0$

(2) If V *is a complex inner product space, then*

$$\langle Tv, v\rangle = 0 \text{ for all } v \in V \quad \Rightarrow \quad T = 0$$

but this does not hold in general for real inner product spaces.

Proof. (1) **HW**(1p)

(2) For $r \in F$ and $x, y \in V$ let $v = rx + y$. Then

$$\begin{aligned} 0 &= \langle T(rx+y), rx+y\rangle \\ &= |r|^2\langle Tx, x\rangle + \langle Ty, y\rangle + r\langle Tx, y\rangle + \overline{r}\langle Ty, x\rangle \\ &= r\langle Tx, y\rangle + \overline{r}\langle Ty, x\rangle. \end{aligned}$$

Setting $r = 1$ and $r = i$, respectively, gives

$$\langle Tx, y\rangle + \langle Ty, x\rangle = 0 \qquad \text{and} \qquad \langle Tx, y\rangle - \langle Ty, x\rangle = 0.$$

These two equations imply that $\langle Tx, y\rangle = 0$ for all $x, y \in V$ and from part (1) we conclude that $T = 0$. For the last statement, the linear operator T rotating by 90 degrees in the real $\mathbb{R}^2$ plane has the property that $\langle Tv, v\rangle = 0$ for any $v \in V$.

□

1.2 Norm and distance

Definition 1.7 (Norm)**.** *If* V *is an inner product space, the* **norm** *of* $v \in V$ *is defined by*

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

A vector v *is a* **unit vector** *if* $\|v\| = 1$.

Theorem 1.8 (Properties of the norm)**.**

(1) $\|v\| \geq 0$ *and* $\|v\| = 0$ *if and only if* $v = 0$.

(2) For all $r \in F$ *and* $v \in V$, $\|rv\| = |r|\,\|v\|$.

*(3) (***Cauchy-Schwarz inequality***)*

$$|\langle u, v\rangle| \leq \|u\|\,\|v\| \qquad \forall u, v \in V$$

with equality if and only if one of u *and* v *is a scalar multiple of the other.*

*(4) (***Triangle inequality***)*

$$\|u + v\| \leq \|u\| + \|v\| \qquad \forall u, v \in V$$

with equality if and only if one of u *and* v *is a scalar multiple of the other.*

(5) For all $u, v, x \in V$,

$$\|u - v\| \leq \|u - x\| + \|x - v\|$$

(6) For all $u, v \in V$,

$$\big|\, \|u\| - \|v\| \,\big| \leq \|u - v\|$$

*(7) (***Parallelogram law***)*

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 \qquad u, v \in V.$$

Proof. (3) If either u or v is zero the result follows, so we can assume that $u, v \neq 0$. Then for any scalar $r \in F$,

$$\begin{aligned} 0 &\leq \|u - rv\|^2 \\ &= \langle u - rv, u - rv \rangle \\ &= \langle u, u \rangle - \overline{r}\langle u, v \rangle - r[\langle v, u \rangle - \overline{r}\langle v, v \rangle] \end{aligned}$$

Choosing $\overline{r} = \langle v, u \rangle / \langle v, v \rangle$ makes the value in the square brackets equal to 0 and so

$$0 \leq \langle u, u \rangle - \frac{\langle v, u \rangle \langle u, v \rangle}{\langle v, v \rangle} = \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|v\|^2},$$

which is equivalent to the Cauchy-Schwarz inequality. Furthermore, equality holds if and only if $\|u - rv\|^2 = 0$, that is, if and only if $u - rv = 0$, which means that u and v are scalar multiple of one another.

(4) From the Cauchy-Schwarz inequality,

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &\leq \|u\|^2 + 2\|u\| \, \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

from which the triangle inequality follows. **HW**(1p): Prove the statement about equality.

(6) **HW**(1p)

(7) **HW**(2p)

□

Any vector space V together with a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies properties (1), (2) and (4) of Theorem 1.8, is called a **normed linear space** and the function $\|\cdot\|$ is called a **norm**. Thus, any inner product space is a normed linear space.
The inner product on V can be recovered from the norm. In other words, knowing the length of all vectors in V is equivalent to knowing all inner product of vectors in V.

Theorem 1.9 (Polarization identities)**.**

(1) If V is a real inner product space, then

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right).$$

(2) If V is a complex inner product space, then

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) + \frac{1}{4}i\left(\|u+iv\|^2 - \|u-iv\|^2\right).$$

Definition 1.10 (Distance)**.** *Let V be an inner product space. The* **distance** *$d(u, v)$ between any two vectors u and v in V is*

$$d(u, v) = \|u - v\|.$$

Theorem 1.11 (Properties of distance)**.**

(1) $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$

*(2) (**Symmetry**) $d(u, v) = d(v, u)$*

*(3) (**Triangle inequality**) $d(u, v) \leq d(u, w) + d(w, v)$*

Any nonempty set V with a function $d : V \times V \to \mathbb{R}$ that satisfies the properties of theorem 1.11, is called a **metric space** and the function d is called a **metric** on V. Therefore, we have the following relation between various spaces:

$$\text{inner product spaces} \subset \text{normed spaces} \subset \text{metric spaces}.$$

1.3 Orthogonality

Definition 1.12 (Orthogonality)**.** *Let V be inner product space.*

(1) Two vectors $u, v \in V$ are **orthogonal***, written $u \perp v$, if $\langle u, v\rangle = 0$.*

(2) Two subsets $X, Y \subseteq V$ are **orthogonal***, written $X \perp Y$, if $\langle x, y\rangle = 0$ for every pair of $x \in X$ and $y \in Y$.*

(3) The **orthogonal complement** *of a subset $X \subseteq V$ is the set*

$$X^{\perp} = \{v \in V \mid \ v \perp X\}.$$

Here, $v \perp X$ means $\{v\} \perp X$.

Theorem 1.13. *Let V be an inner product space.*

(1) The orthogonal complement $X^\perp$ of any subset $X \subseteq V$ is a subspace of V.

(2) For any subspace S of V,

$$S \cap S^\perp = \{0\}.$$

Definition 1.14 (Sum of subspaces)**.** *Let S and T be subspaces of V. The* **sum** *$S + T$ is defined by*

$$S + T = \{u + v \mid \ u \in S,\, v \in T\}.$$

More generally, the sum of any collection $\{S_i\}_{i\in K}$ of subspaces is the set of all finite sums of vectors from the union $\bigcup S_i$:

$$\sum_{i\in K} S_i = \left\{ s_1 + \cdots + s_n \mid \ s_j \in \bigcup_{i\in K} S_i \right\}.$$

Definition 1.15 (Direct sum)**.** *A vector space V is the* **direct sum** *of a family $\{S_i\}_{i\in K}$ of subspaces of V, written*

$$V = \bigoplus_{i\in K} S_i,$$

if the following hold:

*(1) (**Sum**) V is the sum of the family: $V = \sum_{i\in K} S_i$*

*(2) (**Independence**)*

$$S_i \cap \left(\sum_{j\neq i} S_j \right) = \{0\}.$$

Definition 1.16 (Orthogonal direct sum)**.** *An inner product space V is the* **orthogonal direct sum** *of subspaces S and T, written $V = S \odot T$, if*

$$V = S \oplus T, \qquad S \perp T.$$

More generally, V is the orthogonal direct sum of subspaces $S_1, \ldots, S_n$, written $V = S_1 \odot \cdots \odot S_n$, if

$$V = S_1 \oplus \cdots \oplus S_n, \qquad S_i \perp S_j \text{ for } i \neq j.$$

Theorem 1.17. *Let V be an inner product space. The following are equivalent:*

(1) $V = S \odot T$

(2) $V = S \oplus T$ and $T = S^\perp$

Proof. If $V = S \odot T$, then by definition $T \subseteq S^\perp$. However, if $v \in S^\perp$, then $v = s + t$ where $s \in S$ and $t \in T$. Then s is orthogonal to both t and v and so s is orthogonal to itself, which implies that $s = 0$ and so $v \in T$. Hence, $T = S^\perp$. The converse is obvious. □

Definition 1.18 (Orthogonal set)**.** *A nonempty set $\mathcal{O} = \{u_i \mid \ i \in K\}$ of vectors in an inner product space is said to be an* **orthogonal set** *if $u_i \perp u_j$ for all $i \neq j \in K$. If, in addition, each vector u_i is a unit vector, then $\mathcal{O}$ is called an* **orthonormal set***.*

Theorem 1.19 (Orthogonality implies linear independence)**.** *An orthogonal set of nonzero vectors in V is linearly independent.*

Proof. If $\mathcal{O} = \{u_i \mid i \in K\}$ is an orthogonal set of nonzero vectors and

$$r_1 u_1 + \cdots + r_n u_n = 0$$

then

$$0 = \langle r_1 u_1 + \cdots + r_n u_n, u_k \rangle = r_k \langle u_k, u_k \rangle, \qquad k = 1, \ldots, n$$

and so $r_k = 0$ for all $k = 1, \ldots, n$. Hence, $\mathcal{O}$ is linearly independent. □

Our next goal is to transform a set of vectors into an orthonormal set. If we already have an orthogonal set, it is simple to transform it into an orthonormal set by **normalizing** every vector v in the set:

$$u = \frac{v}{\|v\|}.$$

To transform a set into an orthogonal one, **Gram-Schmidt orthogonalization** is used.

Theorem 1.20 (Gram-Schmidt augmentation)**.** *Let V be an inner product space and let $\mathcal{O} = \{u_1, \dots, u_n\}$ be an orthogonal set of vectors in V. If v does not belong to the span of $\mathcal{O}$, denoted by $Span\{u_1, \dots, u_n\}$, then there is a nonzero $u \in V$ for which $\{u_1, \dots, u_n, u\}$ is orthogonal and*

$$Span\{u_1, \dots, u_n, u\} = Span\{u_1, \dots, u_n, v\}.$$

In particular,

$$u = v - \sum_{i=1}^{n} r_i u_i, \qquad \text{where} \quad r_i = \begin{cases} 0 & \text{if } u_i = 0 \\ \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle} & \text{if } u_i \neq 0 \end{cases}$$

Proof. It is sufficient to check that the vector u given by the formula in the theorem is orthogonal to all u_i, $i = 1, \dots, n$. □

Theorem 1.21 (Gram-Schmidt orthogonalization process)**.** *Let $\{v_1, v_2, \dots\}$ be a sequence of vectors in an inner product space V. Define a sequence $\{u_1, u_2, \dots\}$ by repeated application of Gram-Schmidt augmentation, that is*

$$\begin{aligned} u_1 &= v_1 \\ u_k &= v_k - \sum_{i=1}^{k-1} r_{k,i} u_i, \qquad \text{where} \quad r_{k,i} = \begin{cases} 0 & \text{if } u_i = 0 \\ \frac{\langle v_k, u_i \rangle}{\langle u_i, u_i \rangle} & \text{if } u_i \neq 0 \end{cases}, \qquad k = 2, 3, \dots \end{aligned} \tag{1.1}$$

Then $\{u_1, u_2, \dots\}$ is an orthogonal sequence with the property

$$Span\{u_1, u_2, \dots, u_n\} = Span\{v_1, v_2, \dots, v_n\} \qquad \text{for all } \ n > 0.$$

Also, $u_k = 0$ if and only if $v_k \in Span\{v_1, \dots, v_{k-1}\}$.

Proof. The statement will be proved if we prove: For every $n \in \mathbb{N}$ the following statements hold

(a) $u_i \perp u_j$ for any $i \neq j \in \{1, 2, \dots, n\}$

(b) $Span\{u_1, u_2, \dots, u_n\} = Span\{v_1, v_2, \dots, v_n\}$

(c) $u_n = 0$ if and only if $v_n \in Span\{v_1, \dots, v_{n-1}\}$.

This can be proved by induction, noticing that it is trivially true for $n = 1$ and that if it is true for n, we need to show the statement for $n + 1$ by confirming that

(a') $u_i \perp u_{n+1}$ for any $i \in \{1, 2, \dots, n\}$

(b') $Span\{u_1, u_2, \dots, u_{n+1}\} = Span\{u_1, u_2, \dots, u_n, v_{n+1}\}$

(c') $u_{n+1} = 0$ if and only if $v_{n+1} \in Span\{u_1, \dots, u_n\}$.

But if $v_{n+1} \notin Span\{u_1, \dots, u_n\}$ then (a') and (b') immediately follow from the Gram-Schmidt augmentation process, where we augment the set $\{u_1, \dots, u_n\}$ by the vector v_{n+1}. Hence we have to only check what happens if $v_{n+1} \in Span\{u_1, \dots, u_n\}$, which means that there are $a_i \in F$ such that

$$v_{n+1} = \sum_{i=1}^{n} a_i u_i.$$

Computing the inner products

$$\langle v_{n+1}, u_i \rangle = \begin{cases} 0 & \text{if } u_i = 0 \\ a_i \langle u_i, u_i \rangle & \text{if } u_i \neq 0 \end{cases},$$

we see that $a_i = r_{n+1,i}$ when $u_i \neq 0$ (if $u_i = 0$ then a_i can be any scalar). It now follows from (1.1) that $u_{n+1} = 0$ and (c') is proved. □

Example Consider the inner product space $\mathbb{R}[x]$ of real polynomials, with inner product defined by

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Applying the Gram-Schmidt process to the sequence $\{1, x, x^2, x^3, \dots\}$ yields

$$\begin{aligned}
u_1(x) &= 1 \\
u_2(x) &= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} \cdot 1 = x \\
u_3(x) &= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} \cdot 1 - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} \cdot x = x^2 - \frac{1}{3} \\
u_4(x) &= x^3 - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} dx} \cdot 1 - \frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx} \cdot x - \frac{\int_{-1}^{1} x^3(x^2 - \frac{1}{3})\,dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2\,dx} \cdot \left(x^2 - \frac{1}{3}\right) = x^3 - \frac{3}{5}x,
\end{aligned}$$

and so on. The polynomials in this sequence are the **Legendre polynomials**.

Example The Gram-Schmidt process applied to a real or complex matrix leads to the factorization of the matrix into a product of a matrix with orthogonal columns and an upper triangular matrix.

For $n \leq m$, let us take a $m \times n$ matrix $A = (v_1 \mid v_2 \mid \cdots \mid v_n)$ with columns v_i. The Gram-Schmidt process applied to these columns gives orthogonal vectors $(u_1 \mid u_2 \mid \cdots \mid u_n)$ for which

$$Span\{u_1, u_2, \dots, u_k\} = Span\{v_1, v_2, \dots, v_k\} \qquad \text{for all } \; k \leq n.$$

Writing (1.1) in matrix terms,

$$(v_1 \mid v_2 \mid \cdots \mid v_n) = (u_1 \mid u_2 \mid \cdots \mid u_n) \begin{pmatrix} 1 & r_{2,1} & \dots & r_{n,1} \\ & 1 & \dots & r_{n,2} \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

Normalizing the columns of the first matrix on the right-hand side by $a_i = \|u_i\|$ or $a_i = 1$ if $u_i = 0$, we get

$$(v_1 \mid v_2 \mid \cdots \mid v_n) = \underbrace{\left(\frac{u_1}{a_1} \;\middle|\; \frac{u_2}{a_2} \;\middle|\; \cdots \;\middle|\; \frac{u_n}{a_n}\right)}_{Q} \underbrace{\begin{pmatrix} a_1 & a_1 r_{2,1} & \dots & a_1 r_{n,1} \\ & a_2 & \dots & a_2 r_{n,2} \\ & & \ddots & \\ & & & a_n \end{pmatrix}}_{R},$$

and so $A = QR$, where the columns of Q are orthonormal (in the sense that some columns may be zero vectors - if the vectors $v_1, \ldots, v_n$ are linearly independent, then columns of Q are nonzero) and R is upper triangular matrix with positive entries on the main diagonal.

If $m = n$ and A is nonsingular, then Q is unitary, that is, it satisfies $Q^*Q = QQ^* = I$ (for real matrices, the word "orthogonal" is used, and it means $Q^TQ = QQ^T = I$). On the other hand, if the columns of A are not linearly independent, we can adjust the factorization so that Q is replaced by a unitary matrix without zero columns. Namely, if a column u_i/a_i is zero, then we may replace this column by any vector as long as we replace the (i,i)-entry a_i in R by 0. Therefore, we can take nonzero columns of Q, extend to an orthonormal basis for the span of the columns of Q and replace the zero columns of Q by the additional members of this orthonormal basis. In this way, Q is replaced by a unitary matrix Q', and R is replaced by an upper triangular matrix R' that has nonnegative entires on the main diagonal.

Theorem 1.22 (QR-factorization)**.** *Let $A \in \mathcal{M}_{m,n}(F)$, where $F = \mathbb{C}$ or $F = \mathbb{R}$. There exists a matrix $Q \in \mathcal{M}_{m,n}(F)$ with orthonormal columns and an upper triangular matrix $R \in \mathcal{M}_n(F)$ with nonnegative real entries on the main diagonal for which*

$$A = QR \qquad \textit{called } \textbf{\textit{QR-factorization}}.$$

Moreover, if $m = n$, then Q is unitary (orthogonal). If A is nonsingular, then R can be chosen to have positive entries on the main diagonal, in which case the factors Q and R are unique. If A is real, then Q and R may be taken to be real.

Proof. We have already justified all statements above except the following two:

- Uniqueness for nonsingular A: Assume $QR = Q_1R_1$, then

  $$Q_1^{-1}Q = R_1R^{-1}$$

  and the right side is upper triangular with nonzero entries on the main diagonal and the left-hand side is unitary. But an upper triangular matrix with positive entries on the main diagonal is unitary if and only if it is the identity matrix (**HW**(2p): prove it), and so $Q_1 = Q$ and $R_1 = R$.

- Q, R are real for A real: All computations of the QR-factorization take place in the real field.

□

We mention two important applications of QR-factorization:

(1) **Solving system of linear equations** $Ax = b$: We can rewrite the system as $QRx = b$, and since $Q^{-1} = Q^*$, we have

$$Rx = Q^*b.$$

This is an upper triangular system, which can be easily solved back by substitution, that is, starting from the bottom and working up. This approach is effective when we need to solve several systems with the same matrix but different right-hand sides.

(2) **Finding approximate values of eigenvalues**: For a $n \times n$ matrix A define a sequence of matrices by the following algorithm:

1. Set $A_0 = A$.
2. Let $A_0 = Q_0R_0$ be the QR-factorization of A_0 and let $A_1 = R_0Q_0$.

3. Once A_k has been defined, let $A_k = Q_k R_k$ be the QR-factorization of A_k and let $A_{k+1} = R_k Q_k$.

Then A_k is unitarily (orthogonally) similar to A, since

$$Q_{k-1} A_k Q_{k-1}^* = Q_{k-1}(R_{k-1}Q_{k-1})Q_{k-1}^* = Q_{k-1}R_{k-1} = A_{k-1}.$$

For complex matrices it can be shown that under certain conditions (such as when the eigenvalues of A have distinct norms), the sequence A_k converges entry-wise to an upper triangular matrix U, which therefore has the eigenvalues of A on its main diagonal (**HW**(3p): prove it). Results can be obtained in the real case as well.

1.4 Bases and projections

Definition 1.23.

- *A maximal orthonormal set in an inner product space V is called a* **Hilbert basis** *for V.*
- *A maximal linearly independent set of vectors in a vector space V is called a* **Hamel basis** *of V.*
- *A* **orthonormal basis** *means an orthonormal Hamel basis.*

A member of a collection of sets is said to be **maximal** *if it cannot be expanded to another member by addition of an element.*

Theorem 1.24. *An orthonormal basis is a Hilbert basis.*

Proof. **HW**(2p) □

The converse is not in general true but holds for finite-dimensional spaces.

Theorem 1.25. *Let V be an inner product space. A finite subset $\mathcal{B} = \{u_1, \dots, u_k\}$ of V is an orthonormal (Hamel) basis for V if and only if it is a Hilbert basis for V.*

Proof. In view of the previous theorem, we need to show only one implication. If $\mathcal{B}$ is a finite maximal orthonormal set and $\mathcal{B} \subset \mathcal{C}$, where $\mathcal{C}$ is linearly independent, then we may apply Gram-Schmidt process to extend $\mathcal{B}$ to a strictly larger orthonormal set, a contradiction to the maximality of $\mathcal{B}$. □

Example We show a counterexample to the statement of the theorem in infinitely dimensional space $V = \ell^2$. Let

$$M = \{e_i = (0, \dots, 0, 1, 0, \dots) \mid \ i = 1, 2, \dots\},$$

where e_i has a 1 in the i-th coordinate and 0's elsewhere. Clearly, M is an orthonormal set. Moreover, it is maximal because if $v = \{x_n\} \in \ell^2$ has the property that $v \perp M$, then

$$x_i = \langle v, e_i \rangle = 0 \qquad \forall i,$$

and so $v = 0$. Hence no nonzero vector $v \notin M$ is orthogonal to M. This shows that M is a Hilbert basis for the inner product space ℓ^2.
On the other hand, the vector space $S = Span\, M$ is the subspace of all sequences in ℓ^2 that have finite support, that is, have only finite number of nonzero terms. Since $S \neq \ell^2$, we see that M is not a Hamel basis for the vector space ℓ^2.

If we have a set of vectors $\mathcal{O} = \{u_1, \dots, u_n\} \subset V$, we can ask how to best approximate a given vector v from the vector space V by these vectors.

- If $\mathcal{O}$ is a *basis of* V, we can exactly write

$$v = r_1 u_1 + \cdots r_n u_n,$$

  where $r_1, \dots, r_n$ can be obtained by solving a system of n linear equations.

- If $\mathcal{O}$ is an *orthonormal basis of* V, we can again exactly write

$$v = r_1 u_1 + \cdots r_n u_n,$$

  but now $r_1, \dots, r_n$ are obtained without any effort from the formula $r_i = \langle v, u_i \rangle$, which follows by taking inner product of both sides in the expansion for v.

- If $\mathcal{O}$ is just an *orthonormal set* (not necessarily a basis), we can still adopt the approach in the last case to write

$$\widehat{v} = \langle v, u_1 \rangle u_1 + \cdots + \langle v, u_n \rangle u_n.$$

  This approximation of v is called the **Fourier expansion** of v and $\langle v, u_i \rangle$ are called **Fourier coefficients** of v with respect to $\mathcal{O}$.

Theorem 1.26. *Let $\mathcal{O} = \{u_1, \dots, u_n\}$ be an orthonormal subset of an inner product space V and let $S = Span\, \mathcal{O}$. Then the vector*

$$\widehat{v} = \langle v, u_1 \rangle u_1 + \cdots + \langle v, u_n \rangle u_n$$

has the following properties:

(1) $\widehat{v}$ is the unique vector $s \in S$ for which $(v - s) \perp S$.

(2) $\widehat{v}$ is the **best approximation** *to v from within S, that is, $\widehat{v}$ is the unique vector in S that is closest to v in the sense*

$$\|v - \widehat{v}\| < \|v - s\| \qquad \forall s \in S,\ s \neq \widehat{v}.$$

(3) **Bessel's inequality** *holds:*

$$\|\widehat{v}\| \leq \|v\| \qquad \forall v \in V.$$

Proof. For part (1), since

$$\langle v - \widehat{v}, u_i \rangle = \langle v, u_i \rangle - \langle \widehat{v}, u_i \rangle = 0,$$

it follows that $v - \widehat{v} \in S^\perp$. Moreover, if $v - s \in S^\perp$ for some $s \in S$, then $s - \widehat{v} \in S$ and

$$s - \widehat{v} = (v - \widehat{v}) - (v - s) \in S^\perp,$$

and so $s = \widehat{v}$.
For part (2), if s is any element in S, then $v - \widehat{v} \in S^\perp$ implies that $(v - \widehat{v}) \perp (\widehat{v} - s)$ and so

$$\|v - s\|^2 = \|v - \widehat{v} + \widehat{v} - s\|^2 = \|v - \widehat{v}\|^2 + \|\widehat{v} - s\|^2.$$

Hence, $\|v - s\|$ is smallest if and only if $s = \widehat{v}$ and the smallest value is $\|v - \widehat{v}\|$.
HW(2p): prove part (3). □

Theorem 1.27 (Projection theorem). *If S is a finite dimensional subspace of an inner product space V, then*

$$V = S \odot S^\perp \qquad \text{and} \qquad \dim(V) = \dim(S) + \dim(S^\perp).$$

In particular, if $v \in V$ then

$$v = \widehat{v} + (v - \widehat{v}) \in S \odot S^\perp,$$

where $\widehat{v}$ is the Fourier expansion of v with respect to an orthonormal basis of S.

Proof. We know that $v - \widehat{v} \in S^\perp$, and so $V = S + S^\perp$. However, since $S \cap S^\perp = \{0\}$, we have $V = S \odot S^\perp$. □

Example In this example we will show that the projection theorem does not in general hold for infinitely dimensional subspaces. It actually holds for an infinitely dimensional subspace if the subspace is complete, which is a property we will define later.
To find such a counterexample, take $V = \ell^2$ and S be the subspace spanned by the vectors

$$e_i = (0, \dots, 0, 1, 0, \dots),$$

where 1 is in the i-th coordinate and 0's elsewhere. If $x = \{x_n\} \in S^\perp$, then $x_i = \langle x, e_i \rangle = 0$ for all i and so $x = 0$. Therefore, $S^\perp = \{0\}$ but $S \odot S^\perp = S \neq \ell^2$.

Theorem 1.28. *Let S be a finite dimensional subspace of an inner product space V. Then*

(1) $S^{\perp\perp} = S$

(2) If $X \subseteq V$ and $\dim(\mathrm{Span}\, X) < \infty$, then $X^{\perp\perp} = \mathrm{Span}\, X$.

Proof. We will prove (1) and leave (2) as **HW**(2p).
It is clear that $S \subseteq S^{\perp\perp}$ because $S \perp S^\perp$. On the other hand, if $v \in S^{\perp\perp}$, then the projection theorem implies that $v = s + s'$, where $s \in S$ and $s' \in S^\perp$. Then s' is orthogonal both to s and to v and so s' is orthogonal to itself. Hence, $s' = 0$ and $v = s \in S$, so $S^{\perp\perp} \subseteq S$. □

Theorem 1.29 (Characterization of orthonormal bases). *Let $\mathcal{O} = \{u_1, \dots, u_n\}$ be an orthonormal subset of an inner product space V and let $S = \mathrm{Span}\, \mathcal{O}$. Then the following statements are equivalent:*

(1) $\mathcal{O}$ is an orthonormal basis for V.

(2) $(\mathrm{Span}\, \mathcal{O})^\perp = \{0\}$.

(3) Every vector is equal to its Fourier expansion, that is, $\widehat{v} = v$ for all $v \in V$.

(4) **Bessel's identity** *holds:*

$$\|\widehat{v}\| = \|v\| \qquad \forall v \in V.$$

(5) **Parseval's identity** *holds:*

$$\langle v, w \rangle = [\widehat{v}]_{\mathcal{O}} \cdot [\widehat{w}]_{\mathcal{O}},$$

where $[\widehat{v}]_{\mathcal{O}} \cdot [\widehat{w}]_{\mathcal{O}} = \langle v, u_1 \rangle \overline{\langle w, u_1 \rangle} + \cdots + \langle v, u_n \rangle \overline{\langle w, u_k \rangle}$

is the standard dot product in F^n.

Proof.

(1) $\Rightarrow$ (2) : If there is some nonzero $v \in (Span\,\mathcal{O})^{\perp}$, then $\mathcal{O} \cup \{\frac{v}{\|v\|}\}$ is orthonormal and so $\mathcal{O}$ is not maximal.

(2) $\Rightarrow$ (1) : If $\mathcal{O}$ is not maximal, there is an orthonormal set $\mathcal{P}$ for which $\mathcal{O} \subset \mathcal{P}$. Then any nonzero $v \in \mathcal{P} \setminus \mathcal{O}$ is in $(Span\,\mathcal{O})^{\perp}$.

HW (3p): prove the remaining implications.

□

1.5 Riesz representation theorem

Definition 1.30. *A function $\sigma : V \to W$ on complex vector spaces V, W is called* **conjugate linear** *if it is additive:*

$$\sigma(v_1 + v_2) = \sigma(v_1) + \sigma(v_2) \qquad v_1, v_2 \in V$$

and satisfies

$$\sigma(rv) = \overline{r}\sigma(v) \qquad \forall r \in \mathbb{C},\ \forall v \in V.$$

A **conjugate isomorphism** *is a bijective conjugate linear map.*

An example of a linear functional on a inner product space V is the inner product $\langle \cdot, x\rangle : V \to F$ with a fixed vector x, that is,

$$\langle \cdot, x\rangle(v) = \langle v, x\rangle.$$

Then the linear map $\tau : V \to V^*$ defined by

$$\tau(x) = \langle \cdot, x\rangle$$

is conjugate linear. Moreover, since $\langle \cdot, x\rangle = \langle \cdot, y\rangle$ implies $x = y$, it follows that τ is injective and therefore a conjugate isomorphism (since V is finite-dimensional).

Remark We review the definition of several terms from above:

- A map $f : X \to Y$ is **injective** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$.
- A map $f : X \to Y$ is **surjective** if for every $y \in Y$ there is $x \in X$ such that $f(x) = y$.
- A map is **bijective** if it is both injective and surjective.
- The **dual space** V^* to a vector space V is the set of all linear functionals $f : V \to F$.

Theorem 1.31 (Riesz representation theorem)**.** *Let V be a finite-dimensional inner product space.*

(1) The map $\tau : V \to V^*$ defined by

$$\tau(x) = \langle \cdot, x\rangle$$

is a conjugate isomorphism. In particular, for each $f \in V^*$, there exists a unique vector $x \in V$ for which $f = \langle \cdot, x\rangle$, that is,

$$f(v) = \langle v, x\rangle \qquad \forall v \in V.$$

We call x the **Riesz vector** *for f and denote it by R_f.*

(2) The map $R : V^* \to V$ defined by

$$R(f) = R_f$$

is also a conjugate isomorphism, being the inverse of τ. We call it the **Riesz map**.

Proof.

(1) We prove that τ is surjective. If $f = 0$, then $R_f = 0$, so let us assume that $f \neq 0$. Then $K = Ker(f)$ has codimension 1 and so

$$V = Span\, w \odot K \qquad \text{for } w \in K^{\perp}.$$

Letting $x = \alpha w$ for $\alpha \in F$, we require that

$$f(v) = \langle v, \alpha w \rangle$$

and since this clearly holds for any $v \in K$, it is sufficient to show that it holds for $v = w$, that is,

$$f(w) = \langle w, \alpha w \rangle = \overline{\alpha} \langle w, w \rangle.$$

Thus, $\alpha = \overline{f(w)}/\|w\|^2$ and

$$R_f = \frac{\overline{f(w)}}{\|w\|^2} w.$$

(2) We have

$$\begin{aligned} \langle v, R_{rf+sg} \rangle &= (rf + sg)(v) \\ &= rf(v) + sg(v) \\ &= \langle v, \overline{r} R_f \rangle + \langle v, \overline{s} R_g \rangle \\ &= \langle v, \overline{r} R_f + \overline{s} R_g \rangle \end{aligned}$$

for all $v \in V$ and so

$$R_{rf+sg} = \overline{r} R_f + \overline{s} R_g.$$

□

Remark If $V = \mathbb{R}^n$, then $R_f = (f(e_1), \ldots, f(e_n))$, where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$.

2 Metric spaces

2.1 Introduction

In this chapter we will prepare for investigating convergence properties in infinite dimensional inner product spaces. For this, the notion of a metric, which we have already defined in Definition 1.10, will be important. A metric space is not an algebraic structure but is rather meant to model the abstract properties of distance.

Definition 2.1. *A* **metric space** *is a pair (M, d), where M is a nonempty set and $d : M \times M \to \mathbb{R}$ is a real-valued function, called a* **metric** *on M, with the following properties.*

*(1) (**Positive definiteness**) For all $x, y \in M$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.*

*(2) (**Symmetry**) $d(x, y) = d(y, x)$ $\forall x, y \in M$.*

*(3) (**Triangle inequality**) $d(x, y) \leq d(x, z) + d(z, y)$ $\forall x, y, z \in M$.*

Example

(1) Any nonempty set M is a metric space under the **discrete metric**

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

(2) The set $\mathbb{R}^n$ is a metric space under the **Euclidean metric** defined for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

One can define different metrics on the set $\mathbb{R}^n$, for example,

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|,$$

which makes it a different metric space.

(3) The set $\mathbb{C}^n$ is a metric space under the **unitary metric**

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}.$$

(4) The set $C[a, b]$ of all real-valued (or complex-valued) continuous functions on $[a, b]$ is a metric space under the metric

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

We refer to this metric as the **sup metric**.

(5) The set $C[a, b]$ of all real-valued (or complex-valued) continuous functions on $[a, b]$ is a metric space under the metric

$$d_1(f, g) = \int_a^b |f(x) - g(x)| \, dx.$$

(6) Next example is on a metric space of sequences. We will denote sequences by boldface italic letters, as in $\boldsymbol{x} = \{x_n\}$. The set $\ell^\infty_{\mathbb{R}}$ of all bounded sequences of real numbers is a metric space under the metric defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sup_n |x_n - y_n|.$$

The set $\ell^\infty_{\mathbb{C}}$ of all bounded complex sequences, with the same metric, is also a metric space. As is customary, we will usually denote both of these spaces by ℓ^∞.

(6) For $p \geq 1$, let ℓ^p be the set of all sequences $\boldsymbol{x} = \{x_n\}$ of real (or complex) numbers for which

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

We define the p-norm of $\boldsymbol{x}$ by

$$\|\boldsymbol{x}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Then ℓ^p is a metric space under the metric

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_p = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}.$$

The proof of the fact that ℓ^p is a metric, is left as an exercise (**HW**, see Problems). Here we introduce two important inequalities related to this metric.
Hölder's inequality: Let $p, q \geq 1$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $\boldsymbol{x} \in \ell^p$ and $\boldsymbol{y} \in \ell^q$, then the product sequence $\boldsymbol{xy} = \{x_n y_n\}$ is in ℓ^1 and

$$\|\boldsymbol{xy}\|_1 \leq \|\boldsymbol{x}\|_p \|\boldsymbol{y}\|_q,$$

that is,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q}.$$

A special case with $p = q = 2$ is the **Cauchy-Schwarz inequality**

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \sqrt{\sum_{n=1}^{\infty} |y_n|^2}.$$

Minkowski's inequality: For $p \geq 1$, if $\boldsymbol{x}, \boldsymbol{y} \in \ell^p$, then the sum $\boldsymbol{x} + \boldsymbol{y} = \{x_n + y_n\}$ is in ℓ^p and

$$\|\boldsymbol{x} + \boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p,$$

that is,

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p}.$$

2.2 Open and closed sets

Definition 2.2. *First, we define the notion of open and closed ball that will be used to define open and closed sets. Let M be a metric space. Let $x_0 \in M$ and let r be a positive real number.*

$$\begin{aligned} \textit{Open ball:} \quad & B(x_0, r) = \{x \in M \mid \ d(x, x_0) < r\} \\ \textit{Closed ball:} \quad & \overline{B}(x_0, r) = \{x \in M \mid \ d(x, x_0) \leq r\} \\ \textit{Sphere:} \quad & S(x_0, r) = \{x \in M \mid \ d(x, x_0) = r\} \end{aligned}$$

A subset S of a metric space M is said to be **open** *if each point of S is the center of an open ball that is contained completely in S. More specifically, S is open if for all $x \in S$, there exists an $r > 0$ such that $B(x, r) \subseteq S$.*
A set $T \subseteq M$ is **closed** *if its complement T^c in M is open.*

It is easy to show that an open ball is an open set and a closed ball is a closed set (**HW**(1p): prove it). If $x \in M$, we refer to any open set S containing x as an **open neighborhood** of x. Then a set is open if and only if it contains an open neighborhood of each of its points.

Example In the metric space $\mathbb{R}$ with the usual Euclidean metric, the open balls are the open intervals

$$B(x_0, r) = (x_0 - r, x_0 + r)$$

and the closed balls are the closed intervals

$$\overline{B}(x_0, r) = [x_0 - r, x_0 + r].$$

However, the "clopen" interval $(a, b]$ is neither open nor closed.

Theorem 2.3. *The collection $\mathcal{O}$ of all open subsets of a metric space M has the following properties:*

(1) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$

(2) If $S, T \in \mathcal{O}$ then $S \cap T \in \mathcal{O}$

(3) If $\{S_i \mid i \in K\}$ is any collection of open sets, then $\bigcup_{i \in K} S_i \in \mathcal{O}$

These three properties form the basis for an axiom system that is designed to generalize notions such as convergence and continuity.

Definition 2.4. *Let X be a nonempty set. A collection $\mathcal{O}$ of subsets of X is called a* **topology** *for X if it has the following properties:*

(1) $\emptyset \in \mathcal{O}$, $X \in \mathcal{O}$

(2) If $S, T \in \mathcal{O}$ then $S \cap T \in \mathcal{O}$

(3) If $\{S_i \mid i \in K\}$ is any collection of sets in $\mathcal{O}$, then $\bigcup_{i \in K} S_i \in \mathcal{O}$

We refer to subsets in $\mathcal{O}$ as **open sets** *and the pair $(X, \mathcal{O})$ as a* **topological space**.

Theorem 2.3 says that open sets in a metric space, as defined above, form a topology for M. This topology is called the **topology induced** by the metric. Topological spaces are the most general setting in which we can define concepts such as convergence and continuity. However, in this lecture we will work only with topologies induced by a metric, and thus we will phrase all statements directly in terms of the metric.

2.3 Convergence in a metric space

Definition 2.5. *A sequence in a metric space M* **converges** *to $x \in M$, written $\{x_n\} \to x$, if*

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

Equivalently, $\{x_n\} \to x$ if for any $\varepsilon > 0$, there exists an $N > 0$ such that

$$n > N \quad \Rightarrow \quad d(x_n, x) < \varepsilon,$$

or equivalently,

$$n > N \quad \Rightarrow \quad x_n \in B(x, \varepsilon).$$

x is called the **limit** *of the sequence $\{x_n\}$.*

In the following theorem we characterize closed sets (and therefore also open sets), using convergence.

Theorem 2.6. *Let M be a metric space. A subset $S \subseteq M$ is closed if and only if whenever $\{x_n\}$ is a sequence in S and $\{x_n\} \to x$, then $x \in S$. In loose terms, a subset S is closed if it is closed under taking of sequential limits.*

Proof. Suppose that S is closed and let $\{x_n\} \to x$, where $x_n \in S$ for all n. Suppose that $x \notin S$. Then since $x \in S^c$ and S^c is open, there exists an $\varepsilon > 0$ for which $x \in B(x, \varepsilon) \subseteq S^c$. But this implies that

$$B(x, \varepsilon) \cap \{x_n\} = \emptyset,$$

which contradicts the fact that $\{x_n\} \to x$. Hence, $x \in S$.
Conversely, suppose that S is closed under the taking of limits. We show that S^c is open. Let $x \in S^c$ and suppose to the contrary that no open ball about x is contained in S^c. Consider the open balls $B(x, \frac{1}{n})$ for all $n \geq 1$. Since none of these balls is contained in S^c, for each n, there is an $x_n \in S \cap B(x, \frac{1}{n})$. It is clear that $\{x_n\} \to x$ and so $x \in S$. But x cannot be in both S and S^c. This contradiction implies that S^c is open. Thus, S is closed. □

2.4 The closure of a set

Definition 2.7. *Let S be any subset of a metric space M. The* **closure** *of S, denoted by $cl(S)$, is the smallest closed set containing S.*

Since the entire space M is closed and since the intersection of any collection of closed sets is closed (HW(2p): prove it), the closure of any set S does exist and is the intersection of all closed sets containing S. The following definition characterizes the closure in another way.

Definition 2.8. *Let S be a nonempty subset of a metric space M. An element $x \in M$ is said to be a* **limit point***, or* **accumulation point***, of S if every open ball centered at x meets S at a point other than x itself. Let us denote the set of all limit points of S by S'.*

The following theorem tells us about the relation between limit points and closures. We will not prove it here since the proof is similar to that presented in Calculus classes.

Theorem 2.9. *Let S be a nonempty subset of a metric space M.*

(1) $x \in S'$ if and only if there is a sequence $\{x_n\}$ in S for which $x_n \neq x$ for all n and $\{x_n\} \to x$.

(2) S is closed if and only if $S' \subseteq S$, i.e., if and only if it contains all of its limits points.

(3) $cl(S) = S \cup S'$.

(4) $x \in cl(S)$ if and only if there is a sequence $\{x_n\}$ in S for which $\{x_n\} \to x$.

2.5 Dense subsets

The concept of density is meant to convey the idea of a subset $S \subseteq M$ being "arbitrarily close" to every point in M.

Definition 2.10. *A subset S of a metric space is* **dense** *in M if $cl(S) = M$. A metric space is said to be* **separable** *if it contains a countable dense subset.*

Thus, a subset S of M is dense if every open ball about any point $x \in M$ contains at least one point of S. Every metric space contains a dense subset but the question is if there is dense subset that is countable.

Example

(1) The real line $\mathbb{R}$ is separable, since the rational numbers $\mathbb{Q}$ form a countable dense subset. Similarly, $\mathbb{R}^n$ is separable.

(2) The complex plane $\mathbb{C}$ is separable, as is $\mathbb{C}^n$ for any $n \in \mathbb{N}$.

(3) A discrete metric space is separable if and only if it is countable. (**HW**(2p): prove it)

(4) The space ℓ^∞ is not separable. To see this, consider the set S of all binary sequences

$$S = \{\{x_n\} \mid x_i = 0 \text{ or } 1 \text{ for all } i\}.$$

This set is in one-to-one correspondence with the set of all subsets of $\mathbb{N}$ and so is uncountable. Now, each sequence in S is certainly bounded and so lies in ℓ^∞. Moreover, if $\boldsymbol{x} \neq \boldsymbol{y}$ for two sequences $\boldsymbol{x}, \boldsymbol{y} \in S$, then the two sequences must differ in at least one position and so $d(\boldsymbol{x}, \boldsymbol{y}) = 1$.

in other words, we have a subset S of ℓ^∞ that is uncountable and for which the distance between any two distinct elements is 1. This implies that the balls in the uncountable collection $\{B(s, \frac{1}{3}) \mid s \in S\}$ are mutually disjoint. Hence, no countable set can meet every ball, which implies that no countable set can be dense in ℓ^∞.

(5) The metric spaces ℓ^p are separable for $p \geq 1$. The set S of all sequences of the form

$$s = (q_1, \cdot, q_n, 0, \dots)$$

for all $n > 0$, where the q_i's are rational, is a countable set. Let us show that it is dense in ℓ^p. Any $x \in \ell^p$ satisfies

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

Hence, for any $\varepsilon > 0$, there exists an N such that

$$\sum_{n=N+1}^{\infty} |x_n|^p < \frac{\varepsilon}{2}.$$

Since the rational numbers are dense in $\mathbb{R}$, we can find rational numbers q_i, for which

$$|x_i - q_i|^p < \frac{\varepsilon}{2N}, \qquad i = 1, \dots, N.$$

Hence, if $s = (q_1, \dots, q_N, 0, \dots)$, then

$$d(x, s)^p = \sum_{n=1}^{N} |x_n - q_n|^p + \sum_{n=N+1}^{\infty} |x_n|^p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that there is an element of S arbitrarily close to any element of ℓ^p. Thus S is dense in ℓ^p and so ℓ^p is separable.

2.6 Continuity

In this section we define the notion of continuity of functions, which extends the notion of continuity of functions of several real variables.

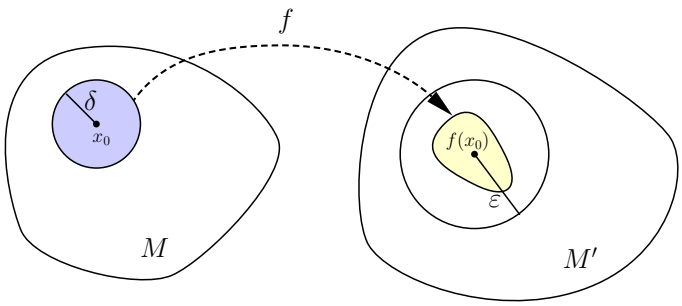
Definition 2.11. *Let $f : M \to M'$ be a function from the metric space (M, d) to another metric space (M', d'). We say that f is* **continuous at** $x_0 \in M$ *if for any $\varepsilon > 0$, there is $\delta > 0$ such that*

$$d(x, x_0) < \delta \quad \Rightarrow \quad d'(f(x), f(x_0)) < \varepsilon$$

or, equivalently,

$$f\big(B(x_0, \delta)\big) \subseteq B(f(x_0), \varepsilon).$$

A function is **continuous** *(on M) if it is continuous at every point x_0 of M.*



Next, we look at the relation between the notions of convergence and continuity (corresponding to sequential definition of continuity for functions of several variables).

Theorem 2.12. *A function $f : M \to M'$ is continuous if and only if whenever $\{x_n\}$ is a sequence in M that converges to $x_0 \in M$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$, in short,*

$$x_n \to x_0 \quad \Rightarrow \quad f(x_n) \to f(x_0).$$

Proof.

- Suppose first that f is continuous at x_0 and let $x_n \to x_0$. Then, given $\varepsilon > 0$, there is $\delta > 0$ such that
$$f\big(B(x_0, \delta)\big) \subseteq B(f(x_0), \varepsilon).$$
Since now $x_n \to x_0$, there exists $N > 0$ such that $x_n \in B(x_0, \delta)$ for $n > N$ and so
$$n > N \quad \Rightarrow \quad f(x_n) \in B(f(x_0), \varepsilon).$$
Thus, $f(x_n) \to f(x_0)$.
- Conversely, suppose that $x_n \to x_0$ implies $f(x_n) \to f(x_0)$. We employ the proof by contradiction and assume that f is not continuous at x_0. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$
$$f\big(B(x_0, \delta)\big) \not\subseteq B(f(x_0), \varepsilon).$$
Hence, for all $n > 0$,
$$f\big(B(x_0, \tfrac{1}{n})\big) \not\subseteq B(f(x_0), \varepsilon),$$

and so we may construct a sequence $\{x_n\}$ by choosing each term x_n with the property that

$$x_n \in B(x_0, \tfrac{1}{n}), \qquad \text{but} \quad f(x_n) \notin B\big(f(x_0), \varepsilon\big).$$

This means that $x_n \to x_0$ but $f(x_n)$ does not converge to $f(x_0)$. This contradiction implies that f must be continuous at x_0.

□

The following theorem says that distance is a continuous function.

Theorem 2.13. *Let (M, d) be a metric space. Then*

$$x_n \to x \quad \textit{and} \quad y_n \to y \qquad \Rightarrow \qquad d(x_n, y_n) \to d(x, y).$$

Proof. **HW**(2p) □

2.7 Completeness

Definition 2.14. *A sequence $\{x_n\}$ in a metric space M is a* **Cauchy sequence** *if for any $\varepsilon > 0$ there exists $N > 0$ for which*

$$n, m > N \quad \Rightarrow \quad d(x_n, x_m) < \varepsilon.$$

The notion of Cauchy sequence is defined for functions on $\mathbb{R}^n$ in Calculus classes, where we learnt that convergence of a sequence is equivalent to its being a Cauchy sequence. However, in general metric spaces this is not true:

- It is easily shown using triangle inequality that any convergent sequence is a Cauchy sequence.
- When the converse holds (i.e., when any Cauchy sequence is convergent), the space M is called complete. The property of completeness plays a major role in the theory of inner product spaces. Complete inner product spaces are then called **Hilbert spaces**.

Definition 2.15. *Let M be a metric space.*

(1) M is said to be **complete** *if every Cauchy sequence in M converges to an element in M.*

(2) A subspace S of M is **complete** *if it is complete as a metric space.*

Theorem 2.16. *Let M be a metric space.*

(1) Any complete subspace of M is closed.

(2) If M is complete, then a subspace S of M is complete if and only if it is closed.

Proof. (1) Assume that S is a complete subspace of M. Let $\{x_n\}$ be a sequence in S for which $x_n \to x \in M$. Then $\{x_n\}$ is a Cauchy sequence in S and since S is complete, $\{x_n\}$ must converge to an element of S. Hence, S is closed.

(2) In view of part (1), we need to only prove that if S is closed and $\{x_n\}$ is a Cauchy sequence in S, then $\{x_n\}$ converges in S. Since $\{x_n\}$ is also a Cauchy sequence in the complete space M, it must converge to some $x \in M$. But since S is closed, we have $x \in S$. Hence, S is complete.

□

Example

(1) It is well-known from Calculus classes, though the proof is not simple, that the metric spaces $\mathbb{R}, \mathbb{R}^n, \mathbb{C}, \mathbb{C}^n$ are complete. Let us prove that $\mathbb{R}^n$ is complete provided we know that $\mathbb{R}$ is complete. Suppose $\{x_k\}$ is a Cauchy sequence in $\mathbb{R}^n$, where

$$x_k = (x_{k,1}, \ldots, x_{k,n}).$$

Thus,

$$d(x_k, x_m)^2 = \sum_{i=1}^{n} (x_{k,i} - x_{m,i})^2 \to 0 \qquad \text{as } k, m \to \infty,$$

and so, for each coordinate position i,

$$(x_{k,i} - x_{m,i})^2 \leq d(x_k, x_m)^2 \to 0,$$

which shows that the sequences $\{x_{k,i}\}_{k=1,2,\ldots}$ of i-th coordinates are all Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we must have

$$x_{k,i} \to y_i \qquad \text{as } k \to \infty.$$

If $y = (y_1, \ldots, y_n)$, then

$$d(x_k, y)^2 = \sum_{i=1}^{n} (x_{k,i} - y_i)^2 \to 0 \qquad \text{as } k \to \infty,$$

and so $x_n \to y \in \mathbb{R}^n$. This proves that $\mathbb{R}^n$ is complete.

(2) The metric space $(C[a,b], d)$ of all real-valued (or complex-valued) continuous functions on $[a,b]$ with the metric

$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

is complete. To see this, we first observe that the limit with respect to d is the uniform limit on $[a,b]$, that is $d(f_n, f) \to 0$ if and only if $\{f_n\}$ converges uniformly to f, in other words, for any $\varepsilon > 0$, there is $N > 0$ for which

$$n > N \quad \Rightarrow \quad |f_n(x) - f(x)| \leq \varepsilon \qquad \text{for all } x \in [a,b].$$

Now let $\{f_n\}$ be a Cauchy sequence in $(C[a,b], d)$. Thus, for any $\varepsilon > 0$, there is N for which

$$m, n > N \quad \Rightarrow \quad |f_n(x) - f_m(x)| \leq \varepsilon \qquad \text{for all } x \in [a,b]. \tag{2.1}$$

This implies that, for each $x \in [a,b]$, the sequence $\{f(x_n)\}$ is a Cauchy sequence of real (or complex) numbers and so it converges. We can therefore define a function f on $[a,b]$ by

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Letting $m \to \infty$ in (2.1), we get

$$n > N \quad \Rightarrow \quad |f_n(x) - f(x)| \leq \varepsilon \qquad \text{for all } x \in [a,b].$$

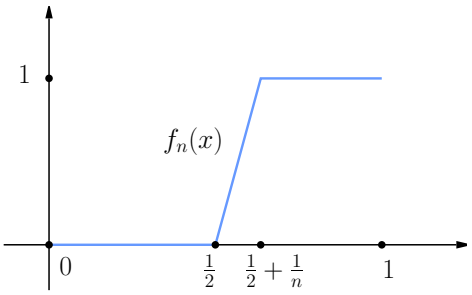
Thus, $\{f_n\}$ converges to f uniformly. It is well known that the uniform limit of continuous functions is continuous and so $f \in C[a,b]$. Thus $(C[a,b], d)$ is complete.

(3) The metric space $(C[a,b], d_1)$ of all real-valued (or complex-valued) continuous functions on $[a,b]$ with the metric

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx$$

is not complete. For convenience, we take $[a,b] = [0,1]$ and consider the sequence of functions

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ nx - \frac{n}{2} & \text{if } x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}] \\ 1 & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}.$$



One can show that the sequence $\{f_n\}$ is Cauchy but does not converge in $(C[0,1], d_1)$ (**HW**(3p)) because the limit function is discontinuous.

(4) The metric space ℓ^∞ is complete. To see this, suppose that $\{x_n\}$ is a Cauchy sequence in ℓ^∞, where

$$x_n = (x_{n,1}, x_{n,2}, \dots).$$

Then, for each coordinate position i, we have

$$|x_{n,i} - x_{m,i}| \le \sup_j |x_{n,j} - x_{m,j}| \to 0 \qquad \text{as } m, n \to \infty. \tag{2.2}$$

Hence, for each i, the sequence $\{x_{n,i}\}$ of i-th coordinates is a Cauchy sequence in $\mathbb{R}$ (or in $\mathbb{C}$). Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, we have

$$x_{n,i} \to y_i \qquad \text{as } n \to \infty$$

for some y_i for each coordinate position i. We want to show that $y = \{y_i\} \in \ell^\infty$ and that $x_n \to y$. Letting $m \to \infty$ in (2.2) gives

$$\sup_j |x_{n,j} - y_j| \to 0 \qquad \text{as } n \to \infty \tag{2.3}$$

and so, for some n,

$$|x_{n,j} - y_j| < 1 \qquad \text{for all } j,$$

and so

$$|y_j| < 1 + |x_{n,j}| \qquad \text{for all } j.$$

But since $x_n \in \ell^\infty$, it is a bounded sequence and therefore so is $\{y_j\}$. That is, $y = \{y_j\} \in \ell^\infty$. Moreover, since (2.3) implies that $x_n \to y$, we see that ℓ^∞ is complete.

(5) The metric space ℓ^p is complete. To prove this, let $\{x_n\}$ be a Cauchy sequence in ℓ^p, where

$$x_n = (x_{n,1}, x_{n,2}, \dots).$$

Then, for each coordinate position i, we have

$$|x_{n,i} - x_{m,i}|^p \leq \sum_{j=1}^{\infty} |x_{n,j} - x_{m,j}|^p = d(x_n, x_m)^p \to 0 \qquad \text{as } m, n \to \infty,$$

which shows that for each i, the sequence $\{x_{n,i}\}$ of i-th coordinates is a Cauchy sequence in $\mathbb{R}$ (or in $\mathbb{C}$).Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, we have

$$x_{n,i} \to y_i \qquad \text{as } n \to \infty$$

for some y_i for each coordinate position i. We want to show that $y = \{y_i\} \in \ell^p$ and that $x_n \to y$. To this end, observe that for any $\varepsilon > 0$, there is N for which

$$n, m > N \quad \Rightarrow \quad \sum_{i=1}^{r} |x_{n,i} - x_{m,i}|^p \leq \varepsilon \qquad \text{for all } r > 0.$$

Now let $m \to \infty$ to get

$$n > N \quad \Rightarrow \quad \sum_{i=1}^{r} |x_{n,i} - y_i|^p \leq \varepsilon \qquad \text{for all } r > 0.$$

Letting $r \to \infty$, we get for any $n > N$

$$\sum_{i=1}^{\infty} |x_{n,i} - y_i|^p < \varepsilon,$$

which implies that $\{x_n\} - y \in \ell^p$ and so $y \in \ell^p$. In addition, $x_n \to y$.

Not all metric spaces are complete but there is a method to make them complete if they are not. In order to define this completion process exactly, let us first recall the notion of isometry, which is, roughly speaking, a function preserving distances.

Definition 2.17. *Let (M, d) and (M', d') be metric spaces. A function $f : M \to M'$ is called and* **isometry** *if*

$$d'(f(x), f(y)) = d(x, y) \qquad \textit{for all } x, y \in M.$$

If $f : M \to M'$ is a bijective isometry from M to M', we say that M and M' are **isometric**.

Theorem 2.18. *Let $f : (M, d) \to (M', d')$ be an isometry. Then*

(1) f is injective

(2) f is continuous

(3) $f^{-1} : f(M) \to M$ is also an isometry and hence also continuous.

Proof. (1) We just observe that

$$f(x) = f(y) \quad \Leftrightarrow \quad d'(f(x), f(y)) = 0 \quad \Leftrightarrow \quad d(x, y) = 0 \quad \Leftrightarrow \quad x = y.$$

(2) Let $x_n \to x$ in M. Then

$$d'(f(x_n), f(x)) = d(x_n, x) \to 0 \qquad \text{as } n \to \infty$$

and so $f(x_n) \to f(x)$, which proves that f is continuous.

(3) We have

$$d(f^{-1}(f(x)), f^{-1}(f(y))) = d(x, y) = d'(f(x), f(y)),$$

and so $f^{-1} : f(M) \to M$ is an isometry.

□

Theorem 2.19. *Let (M, d) be any metric space. Then there is a complete metric space (M', d') and an isometry $\tau : M \to \tau(M) \subseteq M'$ for which $\tau(M)$ is dense in M'. The metric space (M', d') is called a* **completion** *of (M, d). Moreover, (M', d') is unique, up to bijective isometry.*

For the details of the proof, see [1], p. 316. Here, we give only a very rough outline of it:

1. Define M' as the set of equivalence classes of Cauchy sequences in M, i.e., one element of M' is the whole class of Cauchy sequences in M which converge to the same limit.
2. Define d' on M' as follows: $d'(x, y) = \lim_{n\to\infty} d(x_n, y_n)$, where $x_n \to x$ and $y_n \to y$.
3. The isometry τ is defined by $\tau(x) =$ equivalence class containing the constant sequence $x, x, x, \ldots$. Show that $\tau(M)$ is dense in M'.
4. Prove that (M', d') is complete.
5. Show that (M', d') is unique by constructing the bijective isometry between M' and any other metric space that fulfills the conditions of the theorem.

3 Hilbert spaces

In this section, we will use our knowledge from the previous section on metric spaces and extend the theorems that we proved in the first section for finite dimensional inner product spaces (especially, Fourier approximation and Riesz representation theorem) to infinite dimensional inner product spaces.

In the following we will study the so-called **Hilbert spaces**, which are inner product spaces that are complete under the metric induced by the inner product, that is, the metric

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}.$$

In case an inner product space is not complete, one can construct its completion just as for any metric space.

Theorem 3.1. *Let V be an inner product space. Then there exists a Hilbert space H and an isometry $\tau : V \to H$ for which τV is dense in H. Moreover, H is unique up to isometric isomorphism.*

Example

(1) One of the most important Hilbert spaces is the space ℓ^2. The inner product is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n.$$

The metric induced by this inner product is

$$d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_2 = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{1/2}$$

which agrees with the definition of the metric space ℓ^2 we gave earlier, that is, the metric in the previous example is induced by this inner product. As we saw there, this inner product space is complete and so it is a Hilbert space. In fact, it is the prototype of all Hilbert spaces, introduced by David Hilbert in 1912, even before the axiomatic definition of Hilbert space was given by John von Neumann in 1927.

(2) The other metric spaces ℓ^p $(p \neq 2)$, with distance given by

$$d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_p = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p} \tag{3.1}$$

are not even inner product spaces. More specifically, there is no inner product whose induced metric is given by (3.1). To see this, observe that any norm that comes from an inner product must satisfy the parallelogram law

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2.$$

But the norm in (3.1) does not satisfy this law. For example, take $\boldsymbol{x} = (1,1,0,\ldots)$ and $\boldsymbol{y} = (1,-1,0,\ldots)$. Then

$$\|\boldsymbol{x}+\boldsymbol{y}\|_p = 2, \quad \|\boldsymbol{x}-\boldsymbol{y}\|_p = 2, \quad \|\boldsymbol{x}\|_p = 2^{1/p}, \quad \|\boldsymbol{y}\|_p = 2^{1/p}.$$

Thus, the left side of the parallelogram law is 8 and the right side is $4{\cdot}2^{2/p}$ which equals 8 if and only if $p = 2$.

(3) Another example of a Hilbert space is $L^2(a,b)$, which is the space of all measurable real-valued functions that are square integrable on the interval (a,b):

$$L^2(a,b) = \left\{ f : (a,b) \to \mathbb{R} \,|\, \int_a^b (f(x))^2 \, dx < \infty \right\}.$$

Its metric and norm follow from the inner product defined by

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

Theorem 3.2.

(1) Any complete subspace of an inner product space is closed.

(2) A subspace of a Hilbert space is a Hilbert space if and only if it is closed.

(3) Any finite-dimensional subspace of an inner product space is closed and complete.

Proof. Parts (1) and (2) are the same as in Theorem 2.16, so let us prove that a finite-dimensional subspace S of an inner product space V is closed. Suppose that $\{x_n\}$ is a sequence in S, $x_n \to x$ and $x \notin S$. Let $\mathcal{B} = \{b_1, \ldots, b_m\}$ be an orthonormal Hamel basis for S . The Fourier expansion

$$s = \sum_{i=1}^{m} \langle x, b_i \rangle \, b_i$$

in S has the property that $x - s \neq 0$ but

$$\langle x - s, b_j \rangle = \langle x, b_j \rangle - \langle s, b_j \rangle = 0.$$

Thus, if we write $y = x - s$ and $y_n = x_n - s \in S$, the sequence $\{y_n\}$, which is in S, converges to a vector y that is orthogonal to S. But this is impossible, because $y_n \perp y$ implies that

$$\|y_n - y\|^2 = \|y_n\|^2 + \|y\|^2 \geq \|y\|^2 \neq 0.$$

This proves that S is closed.
To see that any finite-dimensional subspace S of an inner product space is complete, let us embed S (as an inner product space in its own right) in its completion S'. Then S (or rather an isometric copy of S) is a finite-dimensional subspace of a complete inner product space S' and as such it is closed. However, S is dense in S' and so $S = S'$, which shows that S is complete. □

3.1 Infinite series

Since an inner product space allows both addition of vectors and convergence of sequences, we can define the concept of infinite sums, or infinite series.

Definition 3.3. *Let V be an inner product space. The n-th* **partial sum** *of the sequence $\{x_k\}$ in V is*

$$s_n = x_1 + \cdots + x_n.$$

If the sequence $\{s_n\}$ of partial sums converges to a vector $s \in V$, that is, if

$$\|s_n - s\| \to 0 \qquad \text{as} \quad n \to \infty,$$

then we say that the series $\sum x_n$ **converges** *to* s *and write*

$$\sum_{n=1}^{\infty} x_n = s.$$

Furthermore, a series $\sum x_k$ *is said to be* **absolutely convergent** *if the series* $\sum_{n=1}^{\infty} \|x_k\|$ *converges.*

The key relationship between convergence and absolute convergence is given in the next theorem. Note that completeness is required to guarantee that absolute convergence implies convergence.

Theorem 3.4. *Let* V *be an inner product space. Then* V *is complete if and only if absolute convergence of a series implies convergence.*

Proof. Suppose that V is complete and that $\sum \|x_k\| < \infty$. Then the sequence $\{s_n\}$ of partial sums is a Cauchy sequence, for if $n > m$, we have

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| \to 0.$$

Hence, the sequence $\{s_n\}$ converges, that is, the series $\sum x_k$ converges.
Conversely, suppose that absolute convergence implies convergence and let $\{x_n\}$ be a Cauchy sequence in V. We wish to show that this sequence converges. Since $\{x_n\}$ is a Cauchy sequence, for each $k > 0$, there exists an N_k with the property that

$$i, j \geq N_k \Rightarrow \|x_i - x_j\| < \frac{1}{2^k}.$$

Clearly, we can choose $N_1 < N_2 < \cdots$, in which case

$$\left\|x_{N_{k+1}} - x_{N_k}\right\| < \frac{1}{2^k},$$

and so

$$\sum_{k=1}^{\infty} \left\|x_{N_{k+1}} - x_{N_k}\right\| \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

Thus, according to hypothesis, the series

$$\sum_{k=1}^{\infty} \left(x_{N_{k+1}} - x_{N_k}\right)$$

converges. But this is a telescoping series, whose n-th partial sum is

$$x_{N_{n+1}} - x_{N_1}$$

and so the subsequence $\{x_{N_k}\}$ converges. Since any Cauchy sequence that has a convergent subsequence must itself converge, the sequence $\{x_k\}$ converges and so V is complete. □

Example Let us show two counterexamples for spaces that are not complete.

(1) Take $(\mathbb{Q}, |\cdot|)$, i.e., the rationals with absolute value as distance. This space is not complete since the closure of $\mathbb{Q}$ is $\mathbb{R}$. The goal is to find a series of rational numbers which absolutely converges to a rational number but converges to an irrational number. To this end, look at the Taylor expansion of $\sqrt{1+x}$ about $x = 0$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

which converges for $x \in [-1, 1]$. The coefficients a_n are rational numbers and they have alternating signs (positive for n odd and negative for n even) as can be readily confirmed by computing the derivatives of $(1+x)^{1/2}$. Setting $x = -1$, we get

$$0 = 1 - a_1 + a_2 - a_3 + \cdots \qquad \Rightarrow \qquad \sum |a_n| = 1,$$

so the series $\sum a_n$ converges absolutely. On the other hand, setting $x = 1$,

$$\sqrt{2} = 1 + a_1 + a_2 + a_3 + \cdots \qquad \Rightarrow \qquad \sum a_n = \sqrt{2} - 1 \notin \mathbb{Q}.$$

(2) Take the space of complex polynomials on the unit circle with sup norm. Then the series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n}$$

is a series of polynomial terms and converges absolutely. Indeed, $|x| = 1$, hence $\sum |x^n/2^n| = \sum 1/2^n = 2$, which is a (constant) polynomial. However, the series itself does not converge because $\sum x^n/2^n = \frac{2}{2-x}$ is not a polynomial.

3.2 Best approximation

Suppose that V is an inner product space and that S is a subset of V. It is of considerable interest to be able to find, for any $x \in V$, a vector in S that is closest to x in the metric induced by the inner product, should such a vector exist. This is the approximation problem for V. Suppose that $x \in V$ and let

$$\delta = \inf_{s \in S} \|x - s\|.$$

By the definition of infimum, there is a sequence $\{s_n\}$ for which

$$\delta_n = \|x - s_n\| \to \delta.$$

Let us see what we can learn about this sequence. First, if we let $y_k = x - s_k$, then according to the parallelogram law,

$$\|y_k + y_j\|^2 + \|y_k - y_j\|^2 = 2\left(\|y_k\|^2 + \|y_j\|^2\right),$$

or

$$\|y_k - y_j\|^2 = 2\left(\|y_k\|^2 + \|y_j\|^2\right) - 4\left\|\frac{y_k + y_j}{2}\right\|^2. \tag{3.2}$$

Now, if the set S is convex, that is, if

$$x, y \in S \quad \Rightarrow \quad rx + (1-r)y \in S \qquad \text{for all} \quad 0 \le r \le 1$$

(in words, S contains the line segment between any two of its points), then $(s_k + s_j)/2 \in S$ and so

$$\left\|\frac{y_k + y_j}{2}\right\| = \left\|x - \frac{s_k + s_j}{2}\right\| \ge \delta.$$

Thus, (3.2) gives

$$\|y_k - y_j\|^2 \leq 2\left(\|y_k\|^2 + |\,\|y_j\|^2\right) - 4\delta^2 \to 0 \qquad \text{as } \; k, j \to \infty.$$

Hence, if S is convex, then the sequence $\{y_n\} = \{x - s_n\}$ is a Cauchy sequence and therefore so is $\{s_n\}$.

If we also require that S be complete, then the Cauchy sequence $\{s_n\}$ converges to a vector $\widehat{x} \in S$ and by the continuity of the norm, we must have $\|x - \widehat{x}\| = \delta$. We summarize our findings in the following theorem.

Theorem 3.5. *Let V be an inner product space and let S be a complete convex subset of V. Then for any $x \in V$, there exists a unique $\widehat{x} \in S$ for which*

$$\|x - \widehat{x}\| = \inf_{s \in S} \|x - s\|$$

The vector $\widehat{x}$ is called the **best approximation** *to x in S.*

Proof. Only the uniqueness remains to be established. Suppose that

$$\|x - \widehat{x}\| = \delta = \left\|x - x'\right\|.$$

Then, by the parallelogram law,

$$\begin{aligned}
\left\|\widehat{x} - x'\right\|^2 &= \left\|\left(x - x'\right) - \left(x - \widehat{x}\right)\right\|^2 \\
&= 2\|x - \widehat{x}\|^2 + 2\left\|x - x'\right\|^2 - \left\|2x - \widehat{x} - x'\right\|^2 \\
&= 2\|x - \widehat{x}\|^2 + 2\left\|x - x'\right\|^2 - 4\left\|x - \frac{\widehat{x} + x'}{2}\right\|^2 \\
&\leq 2\delta^2 + 2\delta^2 - 4\delta^2 = 0
\end{aligned}$$

and so $\widehat{x} = x'$. □

Since any subspace S of an inner product space V is convex, Theorem 3.5 applies to complete subspaces. However, in this case, we can say more.

Theorem 3.6. *Let V be an inner product space and let S be a complete subspace of V. Then for any $x \in V$, the best approximation to x in S is the unique vector $x' \in S$ for which $x - x' \perp S$.*

Proof. Suppose that $x - x' \perp S$, where $x' \in S$. Then for any $s \in S$, we have $x - x' \perp s - x'$ and so

$$\|x - s\|^2 = \|x - x'\|^2 + \|x' - s\|^2 \geq \|x - x'\|^2$$

Hence $x' = \widehat{x}$ is the best approximation to x in S.
Now we need only show that $x - \widehat{x} \perp S$, where $\widehat{x}$ is the best approximation to x in S. For any $s \in S$, we have

$$\begin{aligned}
\|x - rs\|^2 &= \langle x - rs, x - rs\rangle \\
&= \|x\|^2 - \overline{r}\langle x, s\rangle - r\langle s, x\rangle + r\overline{r}\|s\|^2 \\
&= \|x\|^2 + \|s\|^2\left(r\overline{r} - \overline{r}\frac{\langle x, s\rangle}{\|s\|^2} - r\frac{\overline{\langle x, s\rangle}}{\|s\|^2}\right) \\
&= \|x\|^2 + \|s\|^2\left(r - \frac{\langle x, s\rangle}{\|s\|^2}\right)\left(\overline{r} - \frac{\overline{\langle x, s\rangle}}{\|s\|^2}\right) - \frac{|\langle x, s\rangle|^2}{\|s\|^2} \\
&= \|x\|^2 + \|s\|^2\left|r - \frac{\langle x, s\rangle}{\|s\|^2}\right|^2 - \frac{|\langle x, s\rangle|^2}{\|s\|^2}.
\end{aligned}$$

Now, this is smallest when

$$r = r_0 := \frac{\langle x, s\rangle}{\|s\|^2},$$

in which case

$$\|x - r_0 s\|^2 = \|x\|^2 - \frac{|\langle x, s\rangle|^2}{\|s\|^2}.$$

Replacing x by $x - \widehat{x}$ gives

$$\|x - \widehat{x} - r_0 s\|^2 = \|x - \widehat{x}\|^2 - \frac{|\langle x - \widehat{x}, s\rangle|^2}{\|s\|^2}.$$

But $\widehat{x}$ is the best approximation to x in S and since $\widehat{x} - r_0 s \in S$ we must have $\|x - \widehat{x} - r_0 s\|^2 \geq \|x - \widehat{x}\|^2$
Hence,

$$\frac{|\langle x - \widehat{x}, s\rangle|^2}{\|s\|^2} = 0$$

or equivalently,

$$\langle x - \widehat{x}, s\rangle = 0$$

Hence, $x - \widehat{x} \perp S$. □

According to Theorem 3.6, if S is a complete subspace of an inner product space V, then for any $x \in V$, we may write

$$x = \widehat{x} + (x - \widehat{x}),$$

where $\widehat{x} \in S$ and $x - \widehat{x} \in S^\perp$. Hence, $V = S + S^\perp$ and since $S \cap S^\perp = \{0\}$, we also have $V = S \odot S^\perp$. This is the projection theorem for arbitrary inner product spaces.

Theorem 3.7 (The projection theorem). *If S is a complete subspace of an inner product space V, then*

$$V = S \odot S^{\perp}.$$

In particular, if S is a closed subspace of a Hilbert space H, then

$$H = S \odot S^{\perp}.$$

Theorem 3.8. *Let S, T and T' be subspaces of an inner product space V.*

(1) If $V = S \odot T$ then $T = S^{\perp}$.

(2) If $S \odot T = S \odot T'$ then $T = T'$.

Proof. If $V = S \odot T$, then $T \subseteq S^{\perp}$ by definition of orthogonal direct sum. On the other hand, if $z \in S^{\perp}$, then $z = s + t$, for some $s \in S$ and $t \in T$. Hence,

$$0 = \langle z, s \rangle = \langle s, s \rangle + \langle t, s \rangle = \langle s, s \rangle$$

and so $s = 0$, implying that $z = t \in T$. Thus, $S^{\perp} \subseteq T$. Part (2) follows from part (1). □

Let us denote the closure of the span of a set S of vectors by $\mathrm{cspan}(S)$.

Theorem 3.9. *Let H be a Hilbert space.*

(1) If A is a subset of H, then $\mathrm{cspan}(A) = A^{\perp\perp}$.

(2) If S is a subspace of H, then $\mathrm{cl}(S) = S^{\perp\perp}$.

(3) If K is a closed subspace of H, then $K = K^{\perp\perp}$.

Proof. We leave it as **HW**(3p) to show that $[\mathrm{cspan}(A)]^{\perp} = A^{\perp}$. Hence

$$H = \mathrm{cspan}(A) \odot [\mathrm{cspan}(A)]^{\perp} = \mathrm{cspan}(A) \odot A^{\perp}.$$

But since $A^{\perp}$ is closed, we also have

$$H = A^{\perp} \odot A^{\perp\perp},$$

and so by Theorem 3.8, $\mathrm{cspan}(A) = A^{\perp\perp}$. The rest follows easily from part (1). □

In the problems, we provide an example of a closed subspace K of an inner product space V for which $K \neq K^{\perp\perp}$. Hence, we cannot drop the requirement that H be a Hilbert space in Theorem 3.9.

Corollary 3.10. *If A is a subset of a Hilbert space H, then span(A) is dense in H if and only if $A^{\perp} = \{0\}$.*

Proof. As in the previous proof,

$$H = \mathrm{cspan}(A) \odot A^{\perp}$$

and so $A^{\perp} = \{0\}$ if and only if $H = \mathrm{cspan}(A)$. □

We recall **Hilbert basis** for a Hilbert space H is a maximal orthonormal set in H. Zorn's lemma can be used to show that any nontrivial Hilbert space has a Hilbert basis. Moreover, as we will show later, any two Hilbert bases for a Hilbert space have the same cardinality.
Since an orthonormal set $\mathcal{O}$ is maximal if and only if $\mathcal{O}^\perp = \{0\}$, Corollary 3.10 gives the following characterization of Hilbert bases.

Theorem 3.11. *Let $\mathcal{O}$ be an orthonormal subset of a Hilbert space H. The following are equivalent:*

(1) $\mathcal{O}$ is a Hilbert basis.

(2) $\mathcal{O}^\perp = \{0\}$.

(3) $\mathcal{O}$ is a **total subset** *of H, that is, $\operatorname{cspan}(\mathcal{O}) = H$.*

Part (3) of this theorem says that a subset of a Hilbert space is a Hilbert basis if and only if it is a total orthonormal set.

3.3 Fourier expansions

Our next goal is to find an explicit expression for the best approximation to any vector x from within a closed subspace S of a Hilbert space H. We will divide our investigations into three cases, depending on whether S has finite, countably infinite, or uncountable dimension.

3.3.1 The finite-dimensional case

Suppose that $\mathcal{O} = \{u_1, \ldots, u_n\}$ is an orthonormal set in a Hilbert space H. Recall that the Fourier expansion of any $x \in H$, with respect to $\mathcal{O}$, is given by

$$\widehat{x} = \sum_{k=1}^{n} \langle x, u_k \rangle \, u_k$$

where $\langle x, u_k \rangle$ is the Fourier coefficient of x with respect to u_k. Observe that

$$\langle x - \widehat{x}, u_k \rangle = \langle x, u_k \rangle - \langle \widehat{x}, u_k \rangle = 0,$$

and so $x - \widehat{x} \perp \operatorname{span}(\mathcal{O})$. Thus, according to Theorem 3.6, the Fourier expansion $\widehat{x}$ is the best approximation to x in $\operatorname{span}(\mathcal{O})$. Moreover, since $x - \widehat{x} \perp \widehat{x}$, we have

$$\|\widehat{x}\|^2 = \|x\|^2 - \|x - \widehat{x}\|^2 \leq \|x\|^2,$$

and so

$$\|\widehat{x}\| \leq \|x\|$$

with equality if and only if $x = \widehat{x}$, which happens if and only if $x \in \operatorname{span}(\mathcal{O})$.

Theorem 3.12. *Let $\mathcal{O} = \{u_1, \ldots, u_n\}$ be a finite orthonormal set in a Hilbert space H. For any $x \in H$, the Fourier expansion $\widehat{x}$ of x is the best approximation to x in span $(\mathcal{O})$. We also have* **Bessel's inequality**

$$\|\widehat{x}\| \leq \|x\|$$

or equivalently,

$$\sum_{k=1}^{n} |\langle x, u_k \rangle|^2 \leq \|x\|^2 \tag{3.3}$$

with equality if and only if $x \in \operatorname{span}(\mathcal{O})$.

3.3.2 The countably infinite-dimensional case

In the countably infinite case, we will be dealing with infinite sums and so questions of convergence will arise. Thus, we begin with the following.

Theorem 3.13. *Let $\mathcal{O} = \{u_1, u_2, \ldots\}$ be a countably infinite orthonormal set in a Hilbert space H. The series*

$$\sum_{k=1}^{\infty} r_k u_k \tag{3.4}$$

converges in H if and only if the series

$$\sum_{k=1}^{\infty} |r_k|^2 \tag{3.5}$$

converges in $\mathbb{R}$. If these series converge, then they converge unconditionally (that is, any series formed by rearranging the order of the terms also converges). Finally, if the series (3.4) *converges, then*

$$\left\| \sum_{k=1}^{\infty} r_k u_k \right\|^2 = \sum_{k=1}^{\infty} |r_k|^2 .$$

Proof. Denote the partial sums of the first series by s_n and the partial sums of the second series by p_n. Then for $m \leq n$

$$\|s_n - s_m\|^2 = \left\| \sum_{k=m+1}^{n} r_k u_k \right\|^2 = \sum_{k=m+1}^{n} |r_k|^2 = |p_n - p_m| .$$

Hence $\{s_n\}$ is a Cauchy sequence in H if and only if $\{p_n\}$ is a Cauchy sequence in $\mathbb{R}$. Since both H and $\mathbb{R}$ are complete, $\{s_n\}$ converges if and only if $\{p_n\}$ converges.
If the series (3.5) converges, then it converges absolutely and hence unconditionally. We use the fact that a real series converges unconditionally if and only if it converges absolutely. But if (3.5) converges unconditionally, then so does (3.4). The last part of the theorem follows from the continuity of the norm. □

Now let $\mathcal{O} = \{u_1, u_2, \ldots\}$ be a countably infinite orthonormal set in H. The **Fourier expansion** of a vector $x \in H$ is defined to be the sum

$$\widehat{x} = \sum_{k=1}^{\infty} \langle x, u_k \rangle \, u_k. \tag{3.6}$$

To see that this sum converges, observe that for any $n > 0$, (3.3) gives

$$\sum_{k=1}^{n} |\langle x, u_k \rangle|^2 \leq \|x\|^2,$$

and so

$$\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \leq \|x\|^2,$$

which shows that the series on the left converges. Hence, according to Theorem 3.13 the Fourier expansion (3.6) converges unconditionally. Moreover, since the inner product is continuous,

$$\langle x - \widehat{x}, u_k \rangle = \langle x, u_k \rangle - \langle \widehat{x}, u_k \rangle = 0,$$

and so $x - \widehat{x} \in [\mathrm{span}(\mathcal{O})]^\perp = [\mathrm{cspan}(\mathcal{O})]^\perp$. Hence, $\widehat{x}$ is the best approximation to x in $\mathrm{cspan}(\mathcal{O})$. Finally, since $x - \widehat{x} \perp \widehat{x}$, we again have

$$\|\widehat{x}\|^2 = \|x\|^2 - \|x - \widehat{x}\|^2 \leq \|x\|^2,$$

and so

$$\|\widehat{x}\| \leq \|x\|$$

with equality if and only if $x = \widehat{x}$, which happens if and only if $x \in \mathrm{cspan}(\mathcal{O})$. Thus, the following analog of Theorem 3.12 holds.

Theorem 3.14. *Let $\mathcal{O} = \{u_1, u_2, \ldots\}$ be a countably infinite orthonormal set in a Hilbert space H. For any $x \in H$, the Fourier expansion*

$$\widehat{x} = \sum_{k=1}^{\infty} \langle x, u_k \rangle \, u_k$$

of x converges unconditionally and is the best approximation to x in $\mathrm{cspan}(\mathcal{O})$.
We also have **Bessel's inequality**

$$\|\widehat{x}\| \leq \|x\|,$$

or equivalently,

$$\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \leq \|x\|^2,$$

with equality if and only if $x \in \mathrm{cspan}(\mathcal{O})$.

3.3.3 The uncountably infinite-dimensional case

To discuss the case of an arbitrary orthonormal set $\mathcal{O} = \{u_k \mid k \in K\}$, we need to review the concept of the sum of an arbitrary number of terms, since usual series are defined only for countable number of terms.

Definition 3.15. *Let $\mathcal{K} = \{x_k \mid k \in K\}$ be an arbitrary family of vectors in an inner product space V. The sum $\sum_{k \in K} x_k$ is said to converge to a vector $x \in V$ and we write*

$$x = \sum_{k \in K} x_k$$

if for any $\epsilon > 0$, there exists a finite set $S \subseteq K$ for which

$$T \supset S, T \text{ finite} \quad \Rightarrow \quad \left\| \sum_{k \in T} x_k - x \right\| \leq \epsilon.$$

The function

$$S \mapsto \sum_{k \in S} x_k, \qquad S \text{ finite}$$

is a called a **net** in V and the above definition is referred to as the net definition of convergence.

Theorem 3.16. *Let $\mathcal{K} = \{x_k \mid k \in K\}$ be an arbitrary family of vectors in an inner product space V. If the sum*

$$\sum_{k \in K} x_k$$

converges, then at most a countable number of terms x_k can be nonzero.

Theorem 3.17. *Let $\mathcal{K} = \{x_k \mid k \in K\}$ be an arbitrary family of vectors in an inner product space V.*

(1) If the sum $\sum_{k \in K} x_k$ converges, then for any $\epsilon > 0$, there exists a finite set $I \subseteq K$ such that

$$J \cap I = \emptyset, J \text{ finite} \quad \Rightarrow \quad \left| \sum_{k \in J} x_k \right| \leq \epsilon.$$

(2) If V is a Hilbert space, then the converse of (1) also holds.

Theorem 3.18. *Let $\mathcal{O} = \{u_k \mid k \in K\}$ be an arbitrary orthonormal family of vectors in a Hilbert space H. The two series*

$$\sum_{k \in K} r_k u_k \qquad \text{and} \qquad \sum_{k \in K} |r_k|^2$$

converge or diverge together. If these series converge, then

$$\left\| \sum_{k \in K} r_k u_k \right\|^2 = \sum_{k \in K} |r_k|^2$$

Proof. The first series converges if and only if for every $\epsilon > 0$, there exists a finite set $I \subseteq K$ such that

$$J \cap I = \emptyset, J \text{ finite } \Rightarrow \left\| \sum_{k \in J} r_k u_k \right\|^2 \leq \epsilon^2$$

or equivalently,

$$J \cap I = \emptyset, J \text{ finite } \Rightarrow \sum_{k \in J} |r_k|^2 \leq \epsilon^2$$

and this is precisely what it means for the second series to converge. □

The following is a useful characterization of arbitrary sums of nonnegative real terms.

Theorem 3.19. *Let $\{r_k \mid k \in K\}$ be a collection of nonnegative real numbers. Then*

$$\sum_{k \in K} r_k = \sup_{J \subseteq K, J \text{ finite}} \sum_{k \in K} r_k.$$

The following theorem explains the compatibility of the net version and usual version of definition of convergence.

Theorem 3.20. *Let H be a Hilbert space. If $x_k \in H$, then the following are equivalent:*

(1) $\sum_{k\in\mathbb{N}^+} x_k$ *converges to* x *in the sense of nets*

(2) $\sum_{k=1}^{\infty} x_k$ *converges unconditionally to* x

Let us now think about Fourier expansions. Let $\mathcal{O} = \{u_k \mid k \in K\}$ be an arbitrary orthonormal set in a Hilbert space H. Given any $x \in H$, we may apply Theorem 3.12 to all finite subsets of $\mathcal{O}$, to deduce that

$$\sup_{J\subseteq K,\, J \text{ finite}} \sum_{k\in J} |\langle x, u_k\rangle|^2 \leq \|x\|^2$$

and so Theorem 3.19 tells us that the sum

$$\sum_{k\in K} |\langle x, u_k\rangle|^2$$

converges. Hence, according to Theorem 3.18 , the **Fourier expansion**

$$\widehat{x} = \sum_{k\in K} \langle x, u_k\rangle\, u_k$$

of x also converges and

$$\|\widehat{x}\|^2 = \sum_{k\in K} |\langle x, u_k\rangle|^2 .$$

Note that, according to Theorem 3.16, $\widehat{x}$ is a countably infinite sum of terms of the form $\langle x, u_k\rangle\, u_k$ and so is in $\mathrm{cspan}(\mathcal{O})$.
The continuity of infinite sums with respect to the inner product implies that

$$\langle x - \widehat{x}, u_k\rangle = \langle x, u_k\rangle - \langle \widehat{x}, u_k\rangle = 0$$

and so $x-\widehat{x} \in [\mathrm{span}(\mathcal{O})]^\perp = [\mathrm{cspan}(\mathcal{O})]^\perp$. Hence, Theorem 3.5 tells us that $\widehat{x}$ is the best approximation to x in $\mathrm{cspan}(\mathcal{O})$. Finally, since $x - \widehat{x} \perp \widehat{x}$, we again have $\|\widehat{x}\|^2 = \|x\|^2 - \|x - \widehat{x}\|^2 \leq \|x\|^2$ and so

$$\|\widehat{x}\| \leq \|x\|$$

with equality if and only if $x = \widehat{x}$, which happens if and only if $x \in \mathrm{cspan}(\mathcal{O})$. Thus, we arrive at the most general form of a key theorem about Hilbert spaces.

Theorem 3.21. *Let* $\mathcal{O} = \{u_k \mid k \in K\}$ *be an orthonormal family of vectors in a Hilbert space* H*. For any* $x \in H$*, the Fourier expansion*

$$\widehat{x} = \sum_{k\in K} \langle x, u_k\rangle\, u_k$$

of x *converges in* H *and is the unique best approximation to* x *in* $\mathrm{cspan}(\mathcal{O})$*. Moreover, we have* **Bessel's inequality**

$$\|\widehat{x}\| \leq \|x\|,$$

or, equivalently,

$$\sum_{k\in K} |\langle x, u_k\rangle|^2 \leq \|x\|^2$$

with equality if and only if $x \in \mathrm{cspan}(\mathcal{O})$.

3.4 Characterization of Hilbert spaces

Since by Theorem 3.11 an orthonormal set $\mathcal{O} = \{u_k \mid k \in K\}$ in a Hilbert space H is a Hilbert basis if and only if $\operatorname{cspan}(\mathcal{O}) = H$, the last theorem then leads to the following characterization of Hilbert bases.

Theorem 3.22. *Let $\mathcal{O} = \{u_k \mid k \in K\}$ be an orthonormal family in a Hilbert space H. The following are equivalent:*

(1) $\mathcal{O}$ is a Hilbert basis (a maximal orthonormal set)

(2) $\mathcal{O}^\perp = \{0\}$

(3) $\mathcal{O}$ is total (that is, $\operatorname{cspan}(\mathcal{O}) = H$)

(4) $x = \widehat{x}$ for all $x \in H$

(5) Equality holds in Bessel's inequality for all $x \in H$, that is,

$$\|x\| = \|\widehat{x}\|$$

(6) for all $x \in H$ Parseval's identity

$$\langle x, y\rangle = \langle \widehat{x}, \widehat{y}\rangle$$

holds for all $x, y \in H$, that is,

$$\langle x, y\rangle = \sum_{k\in K} \langle x, u_k\rangle \, \overline{\langle y, u_k\rangle}.$$

Proof. Parts (1), (2) and (3) are equivalent by Theorem 3.11. Part (4) implies part (3), since $\widehat{x} \in \operatorname{cspan}(\mathcal{O})$, and (3) implies (4) since the unique best approximation of any $x \in \operatorname{cspan}(\mathcal{O})$ is itself and so $x = \widehat{x}$. Parts (3) and (5) are equivalent by Theorem 3.21. Parseval's identity follows from part (4). Finally, Parseval's identity for $y = x$ implies that equality holds in Bessel's inequality. □

We now wish to show that all Hilbert bases for a Hilbert space H have the same cardinality and so we can define the Hilbert dimension of H to be that cardinality.

Theorem 3.23. *All Hilbert bases for a Hilbert space H have the same cardinality. This cardinality is called the Hilbert dimension of H, which we denote by $\operatorname{hdim}(H)$.*

Proof. If H has a finite Hilbert basis, then that set is also a Hamel basis and so all finite Hilbert bases have size $\dim(H)$. Suppose next that $\mathcal{B} = \{b_k \mid k \in K\}$ and $\mathcal{C} = \{c_j \mid j \in J\}$ are infinite Hilbert bases for H. Then for each b_k, we have

$$b_k = \sum_{j\in J_k} \langle b_k, c_j\rangle \, c_j,$$

where J_k is the countable set $\{j \mid \langle b_k, c_j\rangle \neq 0\}$. Moreover, since no c_j can be orthogonal to every b_k, we have $\bigcup_K J_k = J$. Thus, since each J_k is countable, we have

$$|J| = \left|\bigcup_{k\in K} J_k\right| \leq \aleph_0 |K| = |K|.$$

By symmetry, we also have $|K| \leq |J|$ and so the Schröder-Bernstein theorem implies that $|J| = |K|$. □

In order to give a characterization of Hilbert spaces, let K be any nonempty set and let

$$\ell^2(K) = \left\{ f : K \to \mathbb{C} \,\middle|\, \sum_{k\in K} |f(k)|^2 < \infty \right\}.$$

The functions in $\ell^2(K)$ are referred to as square summable functions. We can also define a real version of this set by replacing $\mathbb{C}$ by $\mathbb{R}$. We define an inner product on $\ell^2(K)$ by

$$\langle f, g\rangle = \sum_{k\in K} f(k)\overline{g(k)}.$$

The proof that $\ell^2(K)$ is a Hilbert space is quite similar to the proof that $\ell^2 = \ell^2(\mathbb{N})$ is a Hilbert space: **HW**(4p). If we define $\delta_k \in \ell^2(K)$ by

$$\delta_k(j) = \delta_{k,j} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases},$$

then the collection

$$\mathcal{O} = \{\delta_k \mid k \in K\}$$

is a Hilbert basis for $\ell^2(K)$, of cardinality $|K|$. To see this, observe that

$$\langle \delta_i, \delta_j\rangle = \sum_{k\in K} \delta_i(k)\overline{\delta_j(k)} = \delta_{i,j},$$

and so $\mathcal{O}$ is orthonormal. Moreover, if $f \in \ell^2(K)$, then $f(k) \neq 0$ for only a countable number of $k \in K$, say $\{k_1, k_2, \ldots\}$. If we define f' by

$$f' = \sum_{i=1}^{\infty} f(k_i)\, \delta_{k_i},$$

then $f' \in \operatorname{cspan}(\mathcal{O})$ and $f'(j) = f(j)$ for all $j \in K$, which implies that $f = f'$. This shows that $\ell^2(K) = \operatorname{cspan}(\mathcal{O})$ and so $\mathcal{O}$ is a total orthonormal set, that is, a Hilbert basis for $\ell^2(K)$.

Now let H be a Hilbert space, with Hilbert basis $\mathcal{B} = \{u_k \mid k \in K\}$. We define a map $\phi : H \to \ell^2(K)$ as follows. Since $\mathcal{B}$ is a Hilbert basis, any $x \in H$ has the form

$$x = \sum_{k\in K} \langle x, u_k\rangle\, u_k.$$

Since the series on the right converges, Theorem 3.18 implies that

$$\sum_{k\in K} |\langle x, u_k\rangle|^2$$

converges. Hence, another application of Theorem 3.18 implies that the following series converges:

$$\phi(x) = \sum_{k\in K} \langle x, u_k\rangle\, \delta_k.$$

It follows that ϕ is linear and it is not hard to see that it is also bijective. Notice that $\phi(u_k) = \delta_k$ and so ϕ takes the Hilbert basis $\mathcal{B}$ for H to the Hilbert basis $\mathcal{O}$ for $\ell^2(K)$.
Notice also that

$$\|\phi(x)\|^2 = \langle \phi(x), \phi(x)\rangle = \sum_{k\in K} |\langle x, u_k\rangle|^2 = \left\| \sum_{k\in K} \langle x, u_k\rangle\, u_k \right\|^2 = \|x\|^2$$

and so ϕ is an isometric isomorphism. We have proved the following theorem.

Theorem 3.24. *If H is a Hilbert space of Hilbert dimension κ and if K is any set of cardinality κ, then H is isometrically isomorphic to $\ell^2(K)$.*

Example For finite cardinality N the only Hilbert space is the space F^N ($\mathbb{R}^N$ or $\mathbb{C}^N$) with the usual Euclidean inner product. For cardinality $\aleph_0$, we have the space $\ell^2(\mathbb{N}) = \ell^2$, the sequence space, and for cardinality $\aleph_1$, we have the Lebesgue space $L^2(\mathbb{R})$ or $L^2(a, b)$ of square integrable functions.

3.5 The Riesz representation theorem

Not all linear functionals on a Hilbert space have the form "take the inner product with.., " as in the finite-dimensional case. To see this, observe that if $y \in H$, then the function

$$f_y(x) = \langle x, y \rangle$$

is certainly a linear functional on H. However, it has a special property. In particular, the Cauchy-Schwarz inequality gives, for all $x \in H$

$$|f_y(x)| = |\langle x, y \rangle| \leq \|x\|\|y\|$$

or, for all $x \neq 0$

$$\frac{|f_y(x)|}{\|x\|} \leq \|y\|.$$

Noticing that equality holds if $x = y$, we have

$$\sup_{x \neq 0} \frac{|f_y(x)|}{\|x\|} = \|y\|.$$

This prompts us to make the following definition, which we do for linear transformations between Hilbert spaces.

Definition 3.25. *Let $\tau : H_1 \to H_2$ be a linear transformation from H_1 to H_2. Then τ is said to be* **bounded** *if*

$$\sup_{x \neq 0} \frac{\|\tau x\|}{\|x\|} < \infty.$$

If the supremum on the left is finite, we denote it by $\|\tau\|$ and call it the **norm** *of τ.*

Example Take the linear functional (check that it is really linear)

$$f : C([0, \pi]) \to \mathbb{R} \qquad \text{defined by} \qquad f(\varphi) = \int_0^\pi \varphi(x) \sin x \, dx,$$

where $C([0, \pi])$ is equipped with the sup norm. To find the norm of f, we estimate

$$|f(\varphi)| \leq \int_0^\pi |\varphi(x)| \sin x \, dx \leq \sup_{x \in [0,\pi]} |\varphi(x)| \int_0^\pi \sin x \, dx = 2\|\varphi\|_{C([0,\pi])}.$$

Thus we can say that $\|f\| \leq 2$ and f is bounded. Moreover, taking the constant function $\varphi_0(x) \equiv 1$, we discover

$$\frac{|f(\varphi_0)|}{\|\varphi_0\|} = \frac{\int_0^\pi \sin x \, dx}{1} = 2,$$

which implies that in fact $\|f\| = 2$.

Of course, if $f : H \to F$ is a bounded linear functional on H, then

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}.$$

The set of all bounded linear functionals on a Hilbert space H is called the **continuous dual space**, or **conjugate space**, of H and denoted by H^*. Note that this differs from the algebraic dual of H, which is the set of all linear functionals on H. In the finite-dimensional case, however, since all linear functionals are bounded (**HW**(3p): prove it), the two concepts agree. Unfortunately, there is no universal agreement on the notation for the algebraic dual versus the continuous dual.

The following theorem gives some simple reformulations of the definition of norm.

Theorem 3.26. *Let $\tau : H_1 \to H_2$ be a bounded linear transformation. Then*

(1) $\|\tau\| = \sup_{\|x\|=1} \|\tau x\|$

(2) $\|\tau\| = \sup_{\|x\|\leq 1} \|\tau x\|$

(3) $\|\tau\| = \inf\{c \in \mathbb{R} \mid \|\tau x\| \leq c\|x\| \text{ for all } x \in H\}$

Proof. **HW**(3p) □

The following theorem explains the importance of bounded linear transformations.

Theorem 3.27. *Let $\tau : H_1 \to H_2$ be a linear transformation. The following are equivalent:*

(1) τ is bounded

(2) τ is continuous at any point $x_0 \in H$

(3) τ is continuous.

Proof. Suppose that τ is bounded. Then

$$\|\tau x - \tau x_0\| = \|\tau(x - x_0)\| \leq \|\tau\| \, \|x - x_0\| \to 0$$

as $x \to x_0$. Hence, τ is continuous at x_0. Thus, (1) implies (2). If (2) holds, then for any $y \in H$, we have

$$\|\tau x - \tau y\| = \|\tau(x - y + x_0) - \tau(x_0)\| \to 0$$

as $x \to y$, since τ is continuous at x_0 and $x - y + x_0 \to x_0$ as $y \to x$. Hence, τ is continuous at any $y \in H$ and (3) holds. Finally, suppose that (3) holds. Thus, τ is continuous at 0 and so there exists a $\delta > 0$ such that

$$\|x\| \leq \delta \Rightarrow \|\tau x\| \leq 1.$$

In particular,

$$\|x\| = \delta \Rightarrow \frac{\|\tau x\|}{\|x\|} \leq \frac{1}{\delta},$$

and so

$$\|x\| = 1 \Rightarrow \|\delta x\| = \delta \Rightarrow \frac{\|\tau(\delta x)\|}{\|\delta x\|} \leq \frac{1}{\delta} \Rightarrow \frac{\|\tau x\|}{\|x\|} \leq \frac{1}{\delta}.$$

Thus, τ is bounded. □

Now we can state and prove the Riesz representation theorem.

Theorem 3.28 (The Riesz representation theorem)**.** *Let H be a Hilbert space. For any bounded linear functional f on H, there is a unique $z_0 \in H$ such that*

$$f(x) = \langle x, z_0 \rangle$$

for all $x \in H$. Moreover, $\|z_0\| = \|f\|$.

Proof. If $f = 0$, we may take $z_0 = 0$, so let us assume that $f \neq 0$. Hence, $K = \ker(f) \neq H$ and since f is continuous, K is closed. Thus $H = K \odot K^\perp$. Now, the first isomorphism theorem, applied to the linear functional $f : H \to F$ implies that $H/K \approx F$ (as vector spaces). In addition, $H/K \approx K^\perp$ and so $K^\perp \approx F$. In particular, $\dim\left(K^\perp\right) = 1$.
For any $z \in K^\perp$, we have

$$x \in K \Rightarrow f(x) = 0 = \langle x, z \rangle.$$

Since $\dim\left(K^\perp\right) = 1$, all we need to do is to find $0 \neq z \in K^\perp$ for which

$$f(z) = \langle z, z \rangle$$

for then $f(rz) = rf(z) = r\langle z, z \rangle = \langle rz, z \rangle$ for all $r \in F$, showing that $f(x) = \langle x, z \rangle$ for $x \in K^\perp$ as well.
But if $0 \neq z \in K^\perp$, then

$$z_0 = \frac{\overline{f(z)}}{\langle z, z \rangle} z$$

has this property, as can be easily checked. The fact that $\|z_0\| = \|f\|$ has already been established. □

4 Fourier series

In this section, we will introduce the Fourier series of a function in $L^2(-\pi,\pi)$. These Fourier series are obtained by constructing a special orthonormal basis for the Hilbert space $L^2(-\pi,\pi)$, which consists of trigonometric functions.

Definition 4.1. *Let λ be Lebesgue measure on $[-\pi,\pi]$. We write dx for $d\lambda(x)$. The space $L^2(-\pi,\pi)$ consists of equivalence classes of measurable functions $f:[-\pi,\pi]\to\mathbb{C}$ such that*

$$\int_{-\pi}^{\pi}|f(x)|^2\,dx<\infty.$$

The norm is defined by

$$\|f\|_{L^2(-\pi,\pi)}=\left(\int_{-\pi}^{\pi}|f(x)|^2\,dx\right)^{1/2}.$$

Remark

(1) The usual convention is that we write $f\in L^2(-\pi,\pi)$ and remember that this means that the equivalence class $[f]$ corresponding to f belongs to $L^2(-\pi,\pi)$. Since functions in the same equivalence class have the same integral, there is no ambiguity in the convention as long as we deal only with integrals. However we should avoid notation like $[f](x_0)$ for a specified point x_0 because we can change the value of a representative for a class $[f]$ on a set of measure zero (a point $\{x_0\}$ has measure zero) without changing the class.

(2) It may seem artificial to consider the interval $[-\pi,\pi]$ in the definition. However, this is natural due to the fact that trigonometric functions $\sin x,\cos x$ are 2π-periodic. We thus have two options: Either consider the functions $\sin x,\cos x$ on the interval $[-\pi,\pi]$ or any other interval of length 2π, or consider an interval $[0,\ell]$ of a general length ℓ and the ℓ-periodic functions $\sin\frac{2\pi}{\ell}x,\cos\frac{2\pi}{\ell}x$. One option can be obtained by an appropriate scaling from the other, so we are free to choose the first option, where the formulas become a little simpler.

Theorem 4.2. *The space $L^2(-\pi,\pi)$ is a Hilbert space (that is, a complete inner product space) with inner product given by*

$$\langle f,g\rangle=\int_{-\pi}^{\pi}f(x)\overline{g(x)}\,dx.$$

From now on, we will sometimes denote the space $L^2(-\pi,\pi)$ by the symbol H. For every $n\in\mathbb{Z}$ we define functions in H by

$$e_n(x)=\frac{1}{\sqrt{2\pi}}e^{inx}\qquad\text{for }\ x\in[-\pi,\pi].$$

Our goal will be to prove the following statement.

Theorem 4.3. *The system $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis in H.*

First note that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal system in H (**HW**(2p): prove it). The idea of the proof that this system is a basis is to take the Fourier series

$$\sum_{n\in\mathbb{Z}}\langle f,e_n\rangle e_n(x),$$

and its partial sums

$$s_N(x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x),$$

and show that $\|s_N - f\|_{L^2(-\pi,\pi)} \to 0$ as $N \to \infty$, that is,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |s_N(x) - f(x)|^2 \, dx = 0.$$

The proof of this fact will be the topic to follow but first let us look at the properties of Fourier series more specifically.

4.1 Computation of Fourier series

Here we consider only real-valued functions that are 2π-periodic. A function $f(x)$ defined on $\mathbb{R}$ is called a **periodic function** if there exists a number $T > 0$ such that

$$f(x+T) = f(x) \qquad \forall x \in \mathbb{R}.$$

The smallest number T for which the above relation holds is called the **period** of f.

Lemma 4.4. *If f is T-periodic continuous function, then*

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)\, dx.$$

Proof. Consider the function

$$F(x) = \int_x^{x+T} f(s)ds.$$

It is clear that

$$F'(x) = f(x+T) - f(x) = 0 \qquad \forall x \in \mathbb{R},$$

since f is T-periodic. Thus, $F(x)$ is a constant function and $F(a) = \int_a^{a+T} f(x)\, dx = \int_0^T f(x)\, dx = F(0)$. □

Definition 4.5. *Let*

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots$$

be a sequence of functions defined on some interval $I \subset \mathbb{R}$.

- *We say that this sequence is* **convergent** *(or* **pointwise convergent***) to a function $f(x)$ on I if for each fixed point $x \in I$ the number sequence $\{f_n(x)\}$ converges to the number $f(x)$.*
- *The sequence of functions $\{f_n\}$ is said to be* **uniformly convergent** *to a function $f(x)$ on I if for each $\varepsilon > 0$ there exists a number N_ε depending on ε only, such that*

$$|f_n(x) - f(x)| \le \varepsilon \qquad \text{for all } \; n \ge N_\varepsilon \quad \text{and} \quad x \in I.$$

- *For a given sequence of functions $\{f_n\}$ the series*

$$\sum_{n=1}^{\infty} f_n(x)$$

is **convergent** *on I to $s(x)$, if the sequence of partial sums $\{s_N(x)\}$ converges pointwise to $s(x)$ on I, where*

$$s_N(x) = f_1(x) + f_2(x) + \ldots + f_N(x).$$

- *If the sequence* $\{s_N(x)\}$ *is uniformly convergent to* $s(x)$ *then we say that the series* $\sum_{n=1}^{\infty} f_n(x)$ *is* **uniformly convergent**.

Lemma 4.6. *If the functions*

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots$$

are continuous on an interval $[a,b]$ *and the series* $\sum_{n=1}^{\infty} f_n(x)$ *is uniformly convergent on* $[a,b]$*, then the sum of the series* $s(x)$ *is a continuous function on* $[a,b]$*. Moreover the series obtained term by term integration of this series is also convergent and*

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx = \int_a^b s(x)\,dx.$$

Lemma 4.7 (Weierstrass Theorem)**.** *If the functions*

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots$$

are continuous on an interval $[a,b]$ *,*

$$|f_n(x)| \leq a_n, \quad \forall x \in [a,b], n = 1,2,\ldots$$

and the series $\sum_{n=1}^{\infty} a_n$ *is convergent then the series* $\sum_{n=1}^{\infty} f_n(x)$ *is uniformly convergent to some function that is continuous on* $[a,b]$ *.*

The following statement is deduced from Theorem 4.2 and holds for $L^2(-\pi,\pi)$ considered as space of either real-valued or complex-valued functions.

Lemma 4.8. *The system*

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\sin 2x, \ldots\right\}$$

is an orthonormal basis in $L^2(-\pi,\pi)$*.*

Proof. **HW**(3p) □

Now, the natural question is which functions $f(x)$ have trigonometric series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big),$$

and if so, how to compute the coefficients $a_0, a_1, \ldots, a_n, \ldots, b_1, b_2, \ldots$? If we prove Lemma 4.8, we will immediately know that for a function $f \in L^2(-\pi,\pi)$, the coefficients are given by the inner products $a_n = \langle f(x), \frac{1}{\sqrt{\pi}}\cos nx\rangle, b_n = \langle f(x), \frac{1}{\sqrt{\pi}}\sin nx\rangle$, though we cannot say whether these coefficients are unique. The following theorem shows this fact without using Lemma 4.8 but under stronger assumptions on f.

Theorem 4.9. *Suppose that* f *is* 2π*-periodic function and*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big), \tag{4.1}$$

where the series is converges uniformly on the real axis. Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n = 0, 1, 2\dots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad n = 1, 2\dots \end{aligned}$$

In the proof, we will use the following result:

Lemma 4.10.

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \\ \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \\ \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx &= 0. \end{aligned}$$

Proof. First consider the case $m \neq n$. Then

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right],$$

so

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x\,dx \\ &= \frac{1}{2(m+n)}\sin(m+n)x\Big|_{-\pi}^{\pi} + \frac{1}{2(m-n)}\sin(m-n)x\Big|_{-\pi}^{\pi} \\ &= 0. \end{aligned}$$

On the other hand, if $m = n$,

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \left[1 + \cos(2nx)\right]\,dx = \pi.$$

For the second equation, we use the formula

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right],$$

and calculate in a similar way.
The last equation can be obtained from a symmetry argument, i.e., $\sin$ is an odd function and $\cos$ is an even function, hence their product is odd and the interval on $(-\pi, \pi)$ becomes zero. □

Proof of Theorem 4.9. Multiply $\cos(mx)$ to (4.1) and integrate over $(-\pi, \pi)$ to obtain

$$\begin{aligned} &\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \\ &\quad = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos(mx)\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos(nx)\cos(mx)\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin(nx)\cos(mx)\,dx \end{aligned}$$

Since we assume the series converges uniformly, we can exchange the order of integration and (infinite) summation and apply the previous lemma to find

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,dx \\
&= a_m\pi,
\end{aligned}$$

which proves the formula for a_m for $m = 0, 1, 2, \ldots$. The proof of the formula for b_m is analogous. □

Example Find the Fourier series corresponding to the function $f(x)$ given on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} \pi + x, & x \in [-\pi, 0] \\ \pi - x, & x \in [0, \pi] \end{cases},$$

assuming that it exists and converges uniformly.

We calculate

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = 2\pi - \pi = \pi \\
a_n &= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)\,dx = 2\int_0^{\pi}\cos(nx)\,dx - \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx \\
&= -\frac{2}{\pi}\int_0^{\pi} x\left(\frac{1}{n}\sin(nx)\right)'\,dx = -\frac{2}{n\pi}[x\sin(nx)]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\sin(nx)\,dx \\
&= \frac{2}{n\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} = \frac{2}{n\pi}\left[\frac{1}{n} - \frac{1}{n}\cos(n\pi)\right] = \frac{2}{n^2\pi}\left(1 - (-1)^n\right).
\end{aligned}$$

Thus $a_n = 0$ if n is an even number, and $a_n = \frac{4}{n^2\pi}$, if n is an odd number. The Fourier series is

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x.$$

However, we have not yet enough knowledge to justify that we can really write "=" in the above formula. We will confirm later that it is true because f is continuous.

Example Using Fourier series find the value of the sum $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$.

4.2 Convergence of Fourier series

First let us state two results on Fourier series: Bessel's inequality and Riemann-Lebesgue lemma. We have proved in Theorem 3.14 that Bessel's inequality holds for any orthonormal set in a Hilbert space. Applying this general result to the orthonormal sets we have here, we obtain the following two results.

Proposition 4.11 (Bessel's inequality 1)**.** *Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{4.2}$$

be the Fourier series of a function $f \in L^2(-\pi, \pi)$. Then the following Bessel's inequality holds:

$$\pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \int_{-\pi}^{\pi} f^2(x)\, dx = \|f\|^2_{L^2(-\pi,\pi)}.$$

Proposition 4.12 (Bessel's inequality 2)**.** *Let*

$$\sum_{n\in\mathbb{Z}} c_n e_n(x) \tag{4.3}$$

be the Fourier series of a function $f \in L^2(-\pi, \pi)$, where $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$. Then the following Bessel's inequality holds:

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \int_{-\pi}^{\pi} |f(x)|^2\, dx = \|f\|^2_{L^2(-\pi,\pi)}.$$

Riemann-Lebesgue lemma says that the Fourier coefficients decrease to zero. We give two versions of this lemma with different assumptions and different proofs.

Theorem 4.13 (Riemann-Lebesgue lemma 1)**.** *If $f(x)$ is a continuous function on $\mathbb{R}$ with Fourier series* (4.2)*, then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Proof. This follows immediately from Bessel's inequality but we give another proof that does not rely on it. Since $\cos\alpha = -\cos(\alpha + \pi)$ we have

$$\pi a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = -\int_{-\pi}^{\pi} f(x)\cos\left[\left(x + \frac{\pi}{n}\right) n\right]\, dx.$$

Making the change of variables $y = x + \frac{\pi}{n}$ and using Lemma 4.4 we obtain

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = -\int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} f\left(y - \frac{\pi}{n}\right)\cos(ny)\, dy = -\int_{-\pi}^{\pi} f\left(y - \frac{\pi}{n}\right)\cos(ny)\, dy.$$

Hence we have

$$\begin{aligned} 2\pi|a_n| &= 2\left|\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx\right| \\ &= \left|\int_{-\pi}^{\pi} \left[f(x) - f\left(x - \frac{\pi}{n}\right)\right]\cos(nx)\, dx\right| \\ &\leq \int_{-\pi}^{\pi} \left|f(x) - f\left(x - \frac{\pi}{n}\right)\right|\, dx \end{aligned}$$

The function f is continuous on $[-\pi, \pi]$ thus it is uniformly continuous on $[-\pi, \pi]$. Therefore, the integral in the right hand side tends to zero as $n \to \infty$. So $a_n \to 0$ as $n \to \infty$. The proof for b_n is similar. □

Notice that the following theorem has weaker assumptions than being an element of $L^2(-\pi, \pi)$. Indeed, $L^2(-\pi, \pi)$ is a strict subset of $L^1(-\pi, \pi)$ as, for example, the function $f(x) = |x|^{-1/2}$ shows.

Theorem 4.14 (Riemann-Lebesgue lemma 2)**.** *Suppose $f \in L^1(-\pi, \pi)$ with Fourier series* (4.3)*. Then $c_n \to 0$ as $n \to \pm\infty$.*

Proof. Assume first that $f \in L^2(-\pi, \pi)$. From Bessel's inequality we have

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \|f\|^2_{L^2(-\pi,\pi)} < \infty.$$

Thus $\sum_{n\in\mathbb{Z}} |c_n|^2$ is convergent and $c_n \to 0$ as $n \to \pm\infty$.

Now let $f \in L^1(-\pi, \pi)$, and for each $N \in \mathbb{N}$ define

$$f_N(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq N \\ 0, & \text{if } |f(x)| > N. \end{cases}$$

Then $f_N \in L^2(-\pi, \pi)$, and $|f(x) - f_N(x)| \to 0$ as $N \to \infty$ for almost all x in $[-\pi, \pi]$. But $|f - f_N| \leq |f|$, so by the dominated convergence theorem we get

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - f_N(x)|\, dx = 0.$$

The idea of the proof is to use the linearity of Fourier coefficients to write

$$c_n(f) = c_n(f - f_N) + c_n(f_N)$$

and estimate each of the two terms on the right-hand side separately. Here we use the symbol $c_n(g)$ for the Fourier coefficients of function g.

Let $\varepsilon > 0$. Choose N large enough so that

$$\int_{-\pi}^{\pi} |f(x) - f_N(x)|\, dx < \varepsilon\sqrt{\frac{\pi}{2}},$$

and then choose N_0 such that $|c_n(f_N)| < \varepsilon/2$ for $|n| \geq N_0$ (by the first part of the proof, applied to f_N). Then

$$|c_n(f - f_N)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f(x) - f_N(x)|\, dx < \frac{\varepsilon}{2}$$

and therefore

$$|c_n(f)| \leq |c_n(f - f_N)| + |c_n(f_N)| < \varepsilon \qquad \text{for} \quad |n| \geq N_0.$$

This proves the theorem. □

Next, let us look at pointwise convergence of Fourier series. For $N \in \mathbb{N}$, the N'th **Dirichlet kernel** is given by the partial series

$$D_N(t) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}$$

for $t \in \mathbb{R}$. The function $t \to e^{it}$ is periodic with period 2π, so the same is true of D_N. Moreover, D_N is continuous. We have that

$$\sum_{n=-N}^{N} e^{int} = e^{-iNt} \sum_{n=0}^{2N} \left(e^{it}\right)^n = \begin{cases} e^{-iNt} \frac{e^{i(2N+1)t}-1}{e^{it}-1}, & \text{if } t \notin 2\pi\mathbb{Z} \\ 2N+1, & \text{if } t \in 2\pi\mathbb{Z} \end{cases}$$

Recalling de Moivre's formula, we have $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. Therefore

$$D_N(t) = \begin{cases} \frac{\sin\left(N+\frac{1}{2}\right)t}{2\pi \sin\frac{1}{2}t}, & \text{if } t \notin 2\pi\mathbb{Z} \\ \frac{2N+1}{2\pi}, & \text{if } t \in 2\pi\mathbb{Z} \end{cases}$$

which shows that D_N is an even function (meaning $D_N(-t) = D_N(t)$ for all t). From the definition of D_N we have

$$\int_{-\pi}^{\pi} D_N(t)dt = 2\int_{0}^{\pi} D_N(t)dt = 1. \tag{4.4}$$

The importance of the Dirichlet kernel lies in the fact that its convolution with f gives the N'th partial sum of the Fourier series of f. Indeed,

$$\begin{aligned} s_N(x) &= \sum_{n=-N}^{N} c_n e_n(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(y)\overline{e^{iny}} e^{inx}\, dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-N}^{N} f(y) e^{in(x-y)}\, dy \\ &= \int_{-\pi}^{\pi} D_N(x-y) f(y)\, dy \end{aligned} \tag{4.5}$$

Theorem 4.15. *Given $f \in L^1(-\pi,\pi)$, let $\overline{f}$ be its extension to a 2π-periodic function on $\mathbb{R}$. Suppose $x \in \mathbb{R}$ and $s \in \mathbb{C}$ are such that the function*

$$g(y) = \frac{\overline{f}\,(x+y) + \overline{f}\,(x-y) - 2s}{y} \qquad \text{for} \quad y \in (0,\infty)$$

is integrable on an interval $(0,\delta]$ for some $\delta > 0$. Then the Fourier series of f is convergent at x with sum $\sum_{n\in\mathbb{Z}} c_n e_n(x) = s$

Proof. Note that the assumption that g is integrable on $(0,\delta]$ for some $\delta > 0$ implies that g is integrable over any interval $(0,a]$ for any $a > \delta$ because $|g(y)| \le \frac{1}{\delta}\left(|\overline{f}(x+y)| + |\overline{f}(x-y)| + 2|s|\right)$ for $y \in [\delta, a]$.

Since D_N is 2π -periodic, (4.5) gives

$$s_N(x) = \int_{-\pi}^{\pi} D_N(y) f(x+y)\, dy.$$

Since the Lebesgue integral is invariant under the transformation $y \to -y$ and since D_N is an even function, we get

$$\begin{aligned} s_N(x) &= \int_{0}^{\pi} D_N(y) f(x+y)\, dy + \int_{-\pi}^{0} D_N(y) f(x+y)\, dy \\ &= \int_{0}^{\pi} D_N(y) f(x+y)\, dy + \int_{0}^{\pi} D_N(-y) f(x-y)\, dy \\ &= \int_{0}^{\pi} D_N(y)\left(f(x+y) + f(x-y)\right) dy. \end{aligned}$$

By (4.4) we can write $s = 2s\int_0^\pi D_N(y)\,dy$, and therefore

$$\begin{aligned} s_N(x) - s &= \int_0^\pi D_N(y)\left(f(x+y) + f(x-y) - 2s\right)\,dy \\ &= \frac{1}{2\pi}\int_0^\pi \sin\left(\left(N+\tfrac{1}{2}\right)y\right) g(y)\frac{y}{\sin\frac{y}{2}}\,dy \end{aligned}$$

Since g is integrable on an interval $(0,\delta]$ with $\delta > 0$ and $y \to \frac{y}{\sin\frac{y}{2}}$ is bounded, and therefore integrable on $(0,\pi]$, we can define a function h in $L^1(-\pi,\pi)$ by setting $h(y) = g(y)\frac{y}{\sin\left(\frac{y}{2}\right)}$ for $y \in (0,\delta]$ and $h(y) = 0$ for $y \in [-\pi, 0]$. Then also

$$h_1(y) = \frac{1}{2i\sqrt{2\pi}}h(y)e^{\frac{i}{2}y} \quad \text{and} \quad h_2(y) = \frac{1}{2i\sqrt{2\pi}}h(y)e^{-\frac{i}{2}y}$$

belong to $L^1(-\pi,\pi)$, so by Riemann-Lebesgue lemma 4.14 we have $c_{-N}(h_1) \to 0$ and $c_N(h_2) \to 0$ for $N \to \infty$. Using that $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ we obtain

$$s_N(x) - s = \frac{1}{2\pi}\int_0^\pi \sin\left(\left(N+\tfrac{1}{2}\right)y\right) h(y)\,dy = c_{-N}(h_1) - c_N(h_2).$$

Hence $s_N(x) \to s$ when $N \to \infty$, as claimed. □

Definition 4.16. *A function f defined on an interval (a,b) is* **piecewise continuous** *if there is a finite set of points $a = t_0 < t_1 < \cdots t_{k-1} < t_k = b$ such that f is continuous at $x \in (a,b)\backslash\{t_1,\ldots,t_{k-1}\}$ and the limits*

$$\begin{aligned} f(t_i+) &= \lim_{\varepsilon\to 0,\varepsilon>0} f(t_i+\varepsilon) \quad \textit{(right limit)} \\ f(t_j-) &= \lim_{\varepsilon\to 0,\varepsilon>0} f(t_j-\varepsilon) \quad \textit{(left limit)} \end{aligned}$$

exist and are finite for all $i = 0,\ldots,k-1$ and all $j = 1,\ldots,k$

In particular only the values $f(a+)$ and $f(b-)$ are known. Note that if we want to extend f to a periodic function f on $\mathbb{R}$ we must define

$$\overline{f}(a-) := f(b-) \qquad \text{and} \qquad \overline{f}(b+) := f(a+).$$

This applies in particular when we start with a piecewise continuous function on $(-\pi,\pi)$ and we consider its 2π -periodic extension $\overline{f}$ to $\mathbb{R}$.

Corollary 4.17. *Suppose f is a 2π -periodic function on $\mathbb{R}$ which is piecewise continuous on $(-\pi,\pi)$, with points of discontinuity $t_0, t_1, \ldots, t_k$ as in the above definition. Assume f has derivatives from left and right at the point $x \in \mathbb{R}$, in the sense that*

$$f'(x+) = \lim_{\varepsilon\to 0,\varepsilon>0} \frac{f(x+\varepsilon) - f(x+)}{\varepsilon} \tag{4.6}$$

$$f'(x-) = \lim_{\varepsilon\to 0,\varepsilon>0} \frac{f(x-\varepsilon) - f(x-)}{\varepsilon} \tag{4.7}$$

both exist and are finite. Then the Fourier series of f converges at the point x and has sum

$$s = \frac{1}{2}\left(f(x+) + f(x-)\right).$$

If (4.6) *and* (4.7) *exist and are finite, and f is continuous at x, then the Fourier series of f is convergent with sum $f(x)$.*

Proof. Since f is continuous and bounded on each of the intervals $(-\pi, t_1), (t_1, t_2), \ldots, (t_{k-1}, \pi)$, its restriction to $[-\pi, \pi]$ is a function in $L^1(-\pi, \pi)$. We want to apply Theorem 4.15, where we denote f both the function in $L^1(-\pi, \pi)$ and its 2π-periodic extension. Take $s = \frac{1}{2}(f(x+) + f(x-))$. Then g from Theorem 4.15 is

$$g(y) = \frac{f(x+y) - f(x+)}{y} + \frac{f(x-y) - f(x-)}{y}$$

By the assumption on f we have

$$\lim_{\varepsilon \to 0, \varepsilon > 0} g(\varepsilon) = f'(x+) + f'(x-) < \infty$$

This shows that g is piecewise continuous, and therefore integrable, on an interval $(0, \delta]$ for some $\delta > 0$. The claim of the Corollary follows from Theorem 4.15. □

In the following, let us investigate under which conditions Fourier series converges uniformly.

Definition 4.18. *A function f on (a, b) is* **piecewise** C^1 *if f and its derivative f' are piecewise continuous. We assume that the finite set of points $a = t_0 < t_1 < \cdots t_{k-1} < t_k = b$ at which there are discontinuities is the same for f and f'.*

For example, the function $f(x) = |x|$ defined on $(-\pi, \pi)$ is piecewise C^1. In fact f is continuous everywhere, and f' has a discontinuity at 0. Another example of piecewise C^1 function is the Heaviside function $H(x)$, which is discontinuous at $x = 0$.

Note that if f is piecewise C^1 on $(-\pi, \pi)$ then its 2π-periodic extension to $\mathbb{R}$ is piecewise C^1 on any open interval.
Moreover, since the Fourier coefficients are defined by integrals, and since the (Lebesgue) integral does not depend on values of f at a finite number of points, we can assign arbitrary values to $f(t_0), \ldots, f(t_k)$; their choice will not affect the Fourier coefficients.

Lemma 4.19. *Let f be a continuous 2π-periodic function on $\mathbb{R}$ which is piecewise C^1 on $[-\pi, \pi]$. Then $c_n(f') = inc_n(f)$ for all $n \in \mathbb{Z}$.*

Proof. Since f' is piecewise continuous on $[-\pi, \pi]$, it is integrable, so $c_n(f')$ is well-defined. Since f is continuous on $[-\pi, \pi]$, partial integration works in the usual form on $[-\pi, \pi]$, in the sense that we do not split up $[-\pi, \pi]$ at points $t_1, \ldots, t_{k-1}$ (this would give additional terms). So we get the claim from

$$\begin{aligned} c_n(f') &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\ &= \left[\frac{1}{\sqrt{2\pi}} f(x) e^{-inx} \right]_{-\pi}^{\pi} + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} inf(x) e^{-inx} dx \\ &= inc_n(f). \end{aligned}$$

□

Theorem 4.20. *Let f be a continuous 2π-periodic function on $\mathbb{R}$ which is piecewise C^1 on $[-\pi, \pi]$. Then the Fourier series of f is uniformly convergent on $\mathbb{R}$ with sum f.*

Proof. Lemma 4.19 implies that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |c_n(f)| &= |c_0(f)| + \sum_{n\in\mathbb{N}} |c_n(f)| + \sum_{n\in\mathbb{N}} |c_{-n}(f)| \\ &= |c_0(f)| + \sum_{n\in\mathbb{N}} \frac{1}{n} \left|c_n\left(f'\right)\right| + \sum_{n\in\mathbb{N}} \frac{1}{n} \left|c_{-n}\left(f'\right)\right|.\end{aligned}$$

Since f' is piecewise continuous, it is in $L^2(-\pi,\pi)$. Then $\sum_{n\in\mathbb{N}} \frac{1}{n} |c_n(f')|$ is the inner-product in $l^2(\mathbb{N})$ of the sequences $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\{|c_n(f')|\}_{n\in\mathbb{N}}$ (this last one is in $l^2(\mathbb{N})$ by Bessel's inequality applied to f'). It follows $\sum_{n\in\mathbb{N}} \frac{1}{n} |c_n(f')| < \infty$ and similarly for $\sum_{n\in\mathbb{N}} \frac{1}{n} |c_{-n}(f')|$. Then the series $\sum_{n\in\mathbb{N}} |c_n(f)|$ is convergent, and it follows that $\sum_{n\in\mathbb{N}} c_n(f)e_n(x)$ is uniformly convergent. Its limit must be f by the last assertion in Corollary 4.17. □

Finally, we prove Theorem 4.3, which said

The system $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis in H.

Proof. Recall that $C_0^\infty((-\pi,\pi))$ is dense in $L^2(-\pi,\pi)$. Then it is enough to prove that

$$C_0^\infty((-\pi,\pi)) \subset \overline{\operatorname{span}\{e_n \mid n\in\mathbb{Z}\}}.$$

Let $f \in C_0^\infty((-\pi,\pi))$, and let $\{s_N\}_{N\in\mathbb{N}}$ be the sequence of partial sums of the Fourier series of f. Then $s_N \in \operatorname{span}\{e_n \mid n\in\mathbb{Z}\}$. Extend f to a 2π-periodic function $\overline{f}$ on $\mathbb{R}$. Note that $\overline{f}$ will be continuous and piecewise C^1. Theorem 4.20 implies that $s_N - \overline{f} \to 0$ uniformly on $\mathbb{R}$ as $N\to\infty$. Restriction to $(-\pi,\pi)$ gives that $s_N - f \to 0$ uniformly as $N\to\infty$.
Let $\varepsilon > 0$. Choose N_0 such that for $N \geq N_0$ we have $|s_N(x) - f(x)| < \varepsilon/\sqrt{2\pi}$ for all $x \in (-\pi,\pi)$. Then

$$\|s_N - f\|_{L^2}^2 = \int_{-\pi}^{\pi} |s_N(x) - f(x)|^2 \, dx < \varepsilon^2$$

for all $N \geq N_0$. This shows $\|s_N - f\|_{L^2} \to 0$ as $N\to\infty$. The claim of the theorem is proved. □

Remark Theorem 4.3 implies that for any $f \in L^2(-\pi,\pi)$ the following identity (Parseval's identity) holds:

$$\sum_{n\in\mathbb{Z}} |c_n(f)|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \, dx.$$

4.3 Solving the heat equation

We consider the problem

$$u_t = a^2 u_{xx}, \qquad x \in (0,\ell), t > 0 \tag{4.8}$$
$$u(x,0) = f(x), \qquad x \in [0,\ell] \tag{4.9}$$
$$u(0,t) = u(\ell,t) = 0, \qquad t \geq 0. \tag{4.10}$$

We assume that the solution of the problem has the form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ are nonzero functions. Substituting into (4.8) we get

$$X(x)T'(t) = a^2 X''(x)T(t).$$

Dividing both sides of the last equality by $a^2 X(x)T(t)$ we obtain

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}. \tag{4.11}$$

Since the left hand side of (4.11) depends only on t and the right hand side depend only on x each side of this equality can only be equal to some constant. Thus

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \qquad \lambda = \text{constant}$$

or

$$T'(t) = -\lambda a^2 T(t) \tag{4.12}$$
$$X''(x) + \lambda X(x) = 0. \tag{4.13}$$

It follows from (4.10) that

$$X(0) = X(\ell) = 0. \tag{4.14}$$

So we have to find the values of λ for which the equation (4.13) has nonzero solution which satisfies (4.14). The values of λ for which (4.13) has nonzero solution satisfying (4.14) are called **eigenvalues** of the problem (4.13)-(4.14), and the corresponding solutions **eigenfunctions**.

- When $\lambda = 0$, the equation has a general solution

  $$X(x) = Ax + B.$$

  This function satisfies (4.14) just for $A = B = 0$. Thus $\lambda = 0$ is not an eigenvalue.

- If $\lambda < 0$ then general solution of (4.13) has the form

  $$X(x) = C_1 e^{\sqrt{|\lambda|}x} + C_2 e^{-\sqrt{|\lambda|}x}.$$

  It is easy to see that this function satisfies (4.14) just when $C_1 = C_2 = 0$. So (4.13), (4.14) has no negative eigenvalues.

- If $\lambda > 0$ then the general solution of (4.13) has the form

  $$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

  Substituting into (4.14) we obtain

  $$X(0) = C_1 = 0, \qquad X(\ell) = C_2 \sin(\sqrt{\lambda}\ell) = 0.$$

The second equality holds for $\sqrt{\lambda}\ell = n\pi, n = \pm 1, \pm 2, \ldots$. Thus the numbers

$$\lambda_n = \frac{n^2\pi^2}{\ell^2}, \qquad n = 1, 2, \ldots$$

are eigenvalues of the problem (4.13), (4.14) and the functions

$$X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right), \qquad n = 1, 2, \ldots$$

are the corresponding eigenfunctions.

It is easy to see that the general solution of (4.12) for $\lambda = \lambda_n$ has the form

$$T_n(t) = D_n e^{-a^2\lambda_n t}, \qquad n = 1, 2, \ldots.$$

Hence for each $n = 1, 2, \ldots$, the function $e^{-a^2\lambda_n t}\sin\left(\frac{n\pi}{\ell}x\right)$ satisfies (4.8), (4.10). Since the equation (4.13) is a linear equation for each N

$$u_N(x,t) = \sum_{n=1}^{N} D_n e^{-a^2\lambda_n t}\sin\left(\frac{n\pi}{\ell}x\right),$$

where $D_n, n = 1, \ldots, N$ are arbitrary constants also satisfies (4.8), (4.10). Next we try to satisfy the initial condition (4.9):

$$u_N(x,0) = \sum_{n=1}^{N} D_n \sin\left(\frac{n\pi}{\ell}x\right) = f(x).$$

We see that the solution of the problem (4.8) - (4.10) has this form just when the initial function is linear combination of functions

$$\sin\sqrt{\lambda_1}x, \ldots., \sin\sqrt{\lambda_n}x, \qquad \lambda_k = \frac{k^2\pi^2}{\ell^2}.$$

Let us consider the series

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-a^2\lambda_n t}\sin\left(\frac{n\pi}{\ell}x\right). \tag{4.15}$$

Let us note that if $\sum_{n=1}^{\infty}|D_n| < \infty$, then this series is uniformly convergent on $[-\ell,\ell]\times[0,T], \forall T > 0$. The function $u(x,t)$ defined by (4.15) satisfies the boundary conditions (4.10) since each term of the series satisfies these conditions. It follows from (4.15) that $u(x,t)$ satisfies the initial condition (4.9)

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{\ell}x\right)$$

if and only if

$$D_n = f_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)\,dx.$$

However, this is possible only if the coefficients of cos in the Fourier series of the function f are all zero. This happens if f is an odd function. Hence, the idea is to extend the function f in an odd way to the interval $[-\ell,\ell]$ and then take its 2ℓ-periodic extension. If $f(0) = 0$ as in the following theorem then this kind of extension is possible and if, moreover, $f(\ell) = 0$ and f is continuous on $(0,\ell)$, then the extended function is continuous on $\mathbb{R}$.

Theorem 4.21. *If $f(x)$ is continuous on $[0,\ell]$, $f'(x)$ is piecewise continuous on $[0,\ell]$, and $f(0) = f(\ell) = 0$, then the function*

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-a^2\lambda_n t} \sin\left(\frac{n\pi}{\ell}x\right) \tag{4.16}$$

satisfies (4.8)-(4.10).

Proof. Since f is piecewise smooth, by Theorem 4.20 the series

$$\sum_{n=1}^{\infty} |f_n|$$

is convergent. Thus Weierstrass M-test implies that the function $u(x,t)$ is continuous on $[0,\ell]\times[0,\infty)$. Denoting the n-th term of the series 4.16 for u by u_n, let us show that the series

$$\sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}(x,t) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}(x,t)$$

are convergent uniformly for $t \geq t_0, x \in [0,\ell]$, where t_0 is an arbitrary positive number. Continuity of f on $[0,\ell]$ implies boundedness of the sequence $\{f_n\}$. So there exists $M > 0$ so that

$$|f_n| \leq M, \qquad n = 1, 2, \ldots.$$

Thus for each $t \geq t_0$ and $x \in [0,\ell]$ we have

$$\left|\frac{\partial u_n}{\partial t}(x,t)\right| = \left|-f_n a^2 \frac{n^2\pi^2}{\ell^2} e^{-\frac{a^2n^2\pi^2}{\ell^2}t} \sin\left(\frac{n\pi}{\ell}x\right)\right| \leq M_1 n^2 e^{-\frac{a^2n^2\pi^2}{\ell^2}t_0},$$

where $M_1 = Ma^2\frac{\pi^2}{\ell^2}$, and

$$\left|\frac{\partial^2 u_n}{\partial x^2}(x,t)\right| = \left|-f_n \frac{n^2\pi^2}{\ell^2} e^{-\frac{a^2n^2\pi^2}{\ell^2}t} \sin\left(\frac{n\pi}{\ell}x\right)\right| \leq n^2 \frac{M_1}{a^2} e^{-\frac{a^2n^2\pi^2}{\ell^2}t_0}.$$

It is easy to see that the series

$$\sum_{n=1}^{\infty} n^2 e^{-\frac{a^2n^2\pi^2}{\ell^2}t_0}$$

is convergent. Therefore due to Weierstrass M-test the function $u(x,t)$ defined by (4.16) is a solution of the problem (4.8)-(4.10). □

Remark

- Uniqueness of solution holds for the heat equation, so the above obtained solution is the unique solution.
- The following types of problems can be solved in a similar way (see Problems).
  - Heat equation with source term: $u_t = a^2 u_{xx} + h(x,t)$.
  - Heat equation with inhomogeneous boundary conditions $u(0,t) = A$, $u(\ell,t) = B$.
  - Heat equation with Neumann boundary conditions: $u_x(0,t) = u_x(\ell,t) = 0$.

Remark Fourier series find applications in many other fields, besides differential equations. For a short explanation on applications to Weierstrass approximation theorem, number theory, derivation of isoperimetric inequality, ergodic theory and complex analysis, refer to the notes [2].

References

[1] Steven Roman, *Advanced Linear Algebra*, Graduate Texts in Mathematics, Springer, 2008.

[2] Alberto Candel, *Notes on Fourier Series*, https://www.csun.edu/∼ac53971/courses/math650/fourier.pdf